3304 Geophysical Fluid Dynamics Notes

Based on the 2014 spring lectures by Prof E R Johnson

The Author(s) has made every effort to copy down all the content on the board during lectures. The Author(s) accepts no responsibility whatsoever for mistakes on the notes nor changes to the syllabus for the current year. The Author(s) highly recommends that the reader attends all lectures, making their own notes and to use this document as a reference only.

MATH 3304: Geophysical Fluid Dynamics

Office hr: Tues 10am
Fri 11-1pm

Chapter 1: Introduction

- Wave equation $u_{tt} = c^2 u_{xx}$ has solution $F(x+ct) + G(x-ct)$
  F & G are determined by initial and boundary condition.

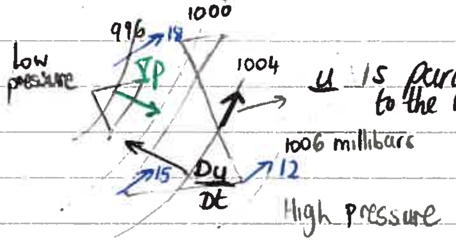
eg modes of a drum

(HW 1)

eg

$$\begin{cases} u = -\operatorname{sgn} x & t = 0 \\ u_t = 0 & t = 0 \end{cases}$$

- Surface isobars
  equal pressure



u is parallel to the isobars

Governing equation:

$$\underbrace{\frac{Du}{Dt}}_{Acc.} = -\frac{1}{\rho}\nabla p \quad \text{— pressure gradient.}$$

* wind speed in knots.

But

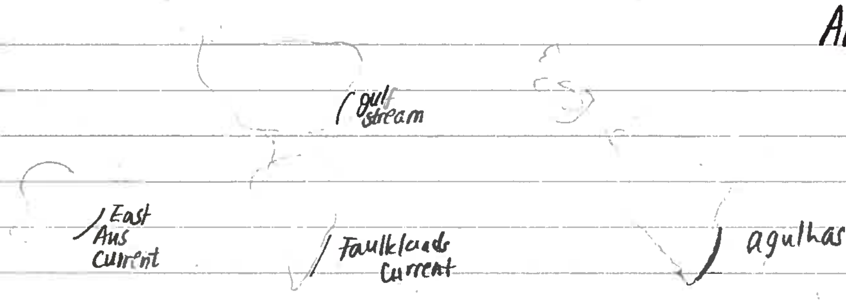
not an inertial frame rotating

$$\Rightarrow \underbrace{\frac{Du}{Dt}}_{\text{This term is negligible}} + \underbrace{2\Omega \wedge u}_{\text{Coriolis}} = -\frac{1}{\rho}\nabla p.$$

Winds follow isobars.
↳ Geostrophic motion.
Earth to turn

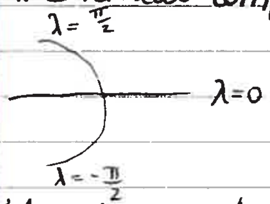
Surface Buys-Ballot - in the Northern Hemisphere, with the wind on your back, high pressure is to your right. (SH to your left).

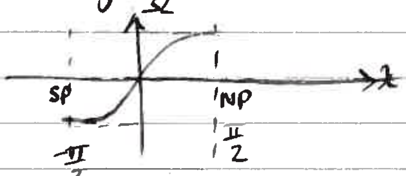
Famous Currents



All these currents are western boundary currents. Why?
↳ If it was because of the rotation of the Earth, then the currents in the southern hemisphere would be to the East.

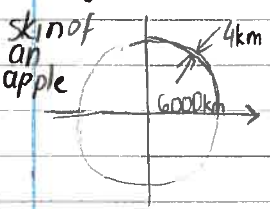
The vertical component of $\underline{\Omega}$ is $\Omega \sin\lambda$, λ is latitude.

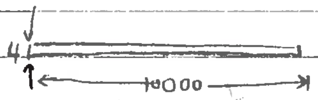




⇒ the vertical component increases away from the equator.

Why do we only look at the vertical component?





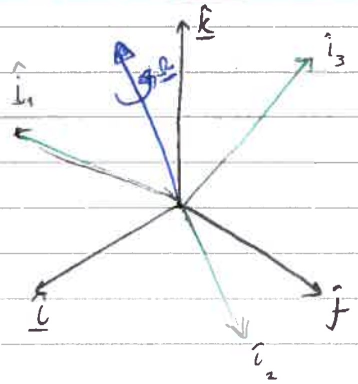
$\underline{u}$ is predominantly horizontal in planetary motion.

$\underline{u} = u\hat{i} + v\hat{j} + w\hat{z}$

$\Rightarrow w \sim 0.1\%$ of u, v.

∴ only vertical part of Ω contributes.

1. The Euler equations in a rotating frame



Inertial frame : I
unit vectors $\hat{i}, \hat{j}, \hat{k}$
Rotating frame : R
unit vectors $\hat{i}_1, \hat{i}_2, \hat{i}_3$
R rotates with any vector $\underline{\Omega}$ relative to I.

$$\left(\frac{d\hat{i}}{dt}\right)_I = 0 \text{ (fixed)}$$

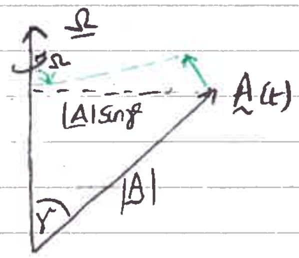
$$\left(\frac{d\hat{i}_1}{dt}\right)_R = 0 \text{ (fixed)}$$

$$\left(\frac{d\hat{i}}{dt}\right)_R \neq 0 \, ?$$

but what are these non zero solutions?

$$\left(\frac{d\hat{i}_1}{dt}\right)_I \neq 0 \, ?$$

Let $\underset{\sim}{A}$ be any vector fixed in R.



$$\left(\frac{dA}{dt}\right)_R = 0$$

$$\left(\frac{dA}{dt}\right)_I = \lim_{\Delta t \to 0} \frac{[A(t+\Delta t) - A(t)]_I}{\Delta t}$$

$A(t+\Delta t) = A(t) + |A|\sin\gamma \cdot |\Omega| \Delta t \, \hat{n}$ where $[\Omega, A, \hat{n}]$ is a RH system ie $(\Omega \wedge A)$.

$$\Rightarrow \left(\frac{dA}{dt}\right)_I = \lim_{\Delta t \to 0} \frac{[A(t+\Delta t) - A(t)]_I}{\Delta t} = \lim \frac{|\Omega||A|\sin\gamma\, \hat{n}\, \Delta t}{\Delta t} = (\Omega \wedge A)$$

Now let $B(t)$ be a variable vector and let its form in R be

$B(t) = B_1(t)\hat{i}_1 + B_2(t)\hat{i}_2 + B_3(t)\hat{i}_3 = B_j(t)\hat{i}_j$ (summation convension)

$$\left(\frac{dB}{dt}\right)_R = \frac{d}{dt}\left(B_j(t)\hat{i}_j\right) = \frac{dB_j(t)}{dt}\hat{i}_j + B_j(t)\underbrace{\left(\frac{d\hat{i}_j}{dt}\right)_R}_{=0}$$

the unit vector in the rotating frame

$$\left(\frac{dB}{dt}\right)_I = \frac{d}{dt}\left(B_j(t)\hat{i}_j(t)\right)$$

$$= \frac{dB_j}{dt}\hat{i}_j(t) + B_j(t)\left(\frac{d\hat{i}_j}{dt}\right)_I$$

constant vector in the rotating frame

$$= \left(\frac{dB}{dt}\right)_R + \Omega \wedge \left(B_j(t)\hat{i}_j\right)$$

$$= \left(\frac{dB}{dt}\right)_R + \Omega \wedge B$$

Example: $\underline{B} = \underline{r}$, the displacement vector of a point

$$\left(\frac{d\underline{r}}{dt}\right)_I = \left(\frac{d\underline{r}}{dt}\right)_R + \underline{\Omega}\wedge\underline{r} \quad \text{or} \quad \underline{u}_I = \underline{u}_R + \underline{\Omega}\wedge\underline{r}$$

$\underline{u}_I$ = vel in I
$\underline{u}_R$ = vel in R.

Example: Acceleration
Need acceleration relative to I to use Newton $\quad (\underline{B} = \underline{u}_I)$

$$\left(\frac{d^2\underline{r}}{dt^2}\right)_I = \left(\frac{d\underline{u}_I}{dt}\right) = \left(\frac{d\underline{u}_I}{dt}\right)_R + \underline{\Omega}\wedge\underline{u}_I$$

$$= \left[\frac{d}{dt}(\underline{u}_R + \underline{\Omega}\wedge\underline{r})\right]_R + \underline{\Omega}\wedge(\underline{u}_R + \underline{\Omega}\wedge\underline{r})$$

$$= \left(\frac{d\underline{u}_R}{dt}\right)_R + \underline{\Omega}\wedge\left(\frac{d\underline{r}}{dt}\right)_R + \underline{\Omega}\wedge\underline{u}_R + \underline{\Omega}\wedge(\underline{\Omega}\wedge\underline{r})$$

(Taking Ω as constant)

what we concentrate on this year.

$$\left(\frac{d\underline{u}_I}{dt}\right)_I = \left(\frac{d\underline{u}_R}{dt}\right)_R + \underbrace{2\underline{\Omega}\wedge\underline{u}_R}_{\text{coriolis acceleration}} + \underbrace{\underline{\Omega}\wedge(\underline{\Omega}\wedge\underline{r})}_{\text{centripetal acceleration}}$$

(acceleration directed towards the centre)

Aside: Physicists (not us): $\underline{F} = m\underline{a}$

In rotating frame $m\underline{a} = \underline{F}$

$$m\left(\frac{d\underline{u}_R}{dt}\right)_R = m\left(\frac{d\underline{u}_I}{dt}\right)_I - 2m\underline{\Omega}\wedge\underline{u}_R - m\underline{\Omega}\wedge(\underline{\Omega}\wedge\underline{r})$$

$$= \underline{F} \underbrace{- \underbrace{2m\underline{\Omega}\wedge\underline{u}_R}_{\text{coriolis force}} - \underbrace{m\underline{\Omega}\wedge(\underline{\Omega}\wedge\underline{r})}_{\text{centrifugal force}}}_{\text{fictitious forces}}$$

force to fight on moving body in rotating frame ← ; → outward force

Recall in an inertial frame $\frac{D\underline{u}_I}{Dt} = -\frac{1}{\rho}\nabla p + \underline{F}$ (Euler)

$\underline{F}$ just a vector ↳ doesn't matter what frame it is in

This in a rotating frame we have $\left(\frac{D\underline{u}_R}{Dt}\right)_R + 2\underline{\Omega}\wedge\underline{u}_R + \underline{\Omega}\wedge(\underline{\Omega}\wedge\underline{r}) = -\frac{1}{\rho}\nabla p + \underline{F}$

Conservation of mass $\rho_t + \nabla\cdot(\rho\underline{u}) = 0 \quad (*)$

- same in R and I, the only time derivative is at ρ, a scalar. (not taking time derivatives of unit vectors $\hat{\imath}_1, \hat{\imath}_2, \hat{\imath}_3$)

ie. $\frac{D\rho}{Dt} + \rho\nabla\cdot\underline{u} = 0$ because $(*) \Rightarrow \rho\nabla\cdot\underline{u} + \underline{u}\cdot\nabla\rho = 0$

filters out sound waves & sound waves need compressibility

We will take both the atmosphere and ocean to be incompressible
This is fine provided the speeds of our flows are small compared to the speed of sound, $[M = \frac{u}{a} \ll 1]$

$\Rightarrow$ A blob of fluid keeps it volume but it keeps its mass
ie it keeps its density $\frac{D\rho}{Dt} = 0 \Rightarrow \boxed{\nabla\cdot\underline{u} = 0}$
↳ but we haven't assume that the density is constant.

Thus we have Euler, $\frac{D\rho}{Dt} = 0$, $\nabla \cdot \underline{u} = 0$ – ie. 5 equations in 5 unknowns.

However for the first half, we will take ρ = constant.

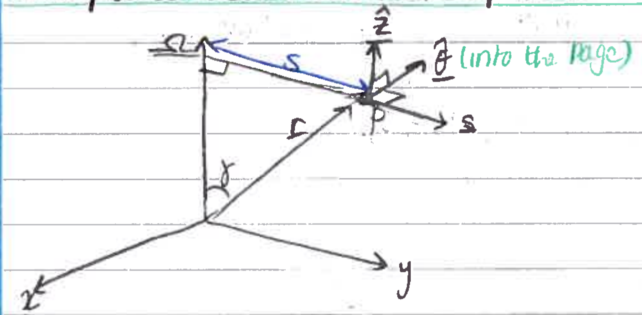
So $\frac{D\rho}{Dt} = 0$ is automatically satisfied.

$$\left(\frac{D\underline{u}_R}{Dt}\right)_R + 2\underline{\Omega} \wedge \underline{u}_R + \underline{\Omega} \wedge (\underline{\Omega} \wedge \underline{r}) = -\frac{1}{\rho}\nabla p + \underline{F}$$

$$\nabla \cdot \underline{u} = 0$$

4 equations & 4 unknowns $\Rightarrow$ closed system.

Centripetal acceleration as a potential



Introduce cylindrical polar coordinates (s, θ, z) as well as usual Cartesians (x, y, z) with $\hat{\underline{z}}$ along $\underline{\Omega}$

$$\underline{\Omega} \wedge \underline{r} = |\underline{\Omega}||\underline{r}| |\sin\gamma| \hat{\underline{\theta}}$$
$$= \Omega s \hat{\underline{\theta}}$$

$$\underline{\Omega} \wedge (\underline{\Omega} \wedge \underline{r}) = \Omega s \, \underline{\Omega} \wedge \hat{\underline{\theta}}$$
$$= +\Omega^2 s \, \hat{\underline{z}} \wedge \hat{\underline{\theta}}$$
$$= -\Omega^2 s \, \hat{\underline{s}}$$

– force directed towards the axis of rotation – centripetal.

Is this a potential, ie does there exist a $G_c(s, \theta, z)$ s.t. $\nabla G_c = -\Omega^2 s \hat{\underline{s}}$

Now in polars, $\nabla G_c = \frac{\partial G_c}{\partial z}\hat{\underline{z}} + \frac{\partial G_c}{\partial s}\hat{\underline{s}} + \frac{1}{s}\frac{\partial G_c}{\partial \theta}\hat{\underline{\theta}}$

so $\frac{\partial G_c}{\partial z} = 0$ $\quad \frac{\partial G_c}{\partial \theta} = 0$ $\quad \frac{\partial G_c}{\partial s} = -\Omega^2 s$

ie. take $G_c = -\frac{1}{2}\Omega^2 s^2$

Now $s = |\hat{\underline{z}} \wedge \underline{r}|$ so $\Omega s = |\underline{\Omega} \wedge \underline{r}|$ so $G_c = -\frac{1}{2}|\underline{\Omega} \wedge \underline{r}|^2$

$(\underline{\Omega} = \Omega \hat{\underline{z}})$

$$\Rightarrow \nabla G_c = \underline{\Omega} \wedge (\underline{\Omega} \wedge \underline{r})$$

From now on, we will always be in the rotating frame R, so drop subscript 'R'.

Now we have $\frac{D\underline{u}}{Dt} + 2\underline{\Omega} \wedge \underline{u} + \nabla G_c = -\frac{1}{\rho}\nabla p + \underline{F}$

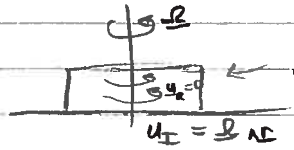
The only external force we will consider is gravity, so $\underline{F} = -g\hat{\underline{z}}$
$= -\nabla G_g$

where $G_g = gz$, gravitational potential per unit mass.

The fluids portal - films on rotating flow.

(i). $\dfrac{D\underline{u}}{Dt} + 2\underline{\Omega}\wedge\underline{u} = -\dfrac{1}{\rho}\nabla p - \nabla G_g - \nabla G_c$

We introduce the equilibrium pressure, p_e, to be the pressure when $\underline{u}=0$
[Notice: $\underline{u}=0$ means $u_R=0$ ie flow is at rest in rotating frame, i.e. solid body rotation relative to the inertial frame.]



← bowl of water

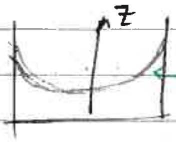
$\underline{u}=0$, then $-\frac{1}{\rho}\nabla p_e - \nabla G_g - \nabla G_c = 0$

i.e. $\nabla(p_e + \rho G_g + \rho G_c) = 0$

Thus w.l.o.g. we can take $p_e = p_0 - \rho G_g - \rho G_c$ $\qquad p_0$ constant.

i.e. $p_e = p_0 \underbrace{- \rho g z}_{\text{hydrostatic pressure } p_h} + \frac{1}{2}\rho\underbrace{|\underline{\Omega}\wedge\underline{r}|^2}_{\Omega^2 s^2 = \Omega^2(x^2+y^2) \text{ and } s \text{ is distance from axis}}$ centripetal term

Notice this can be written $p_e = p_0 - \rho g z + \frac{1}{2}\rho\Omega^2(x^2+y^2)$



this surface has $p_e = p_a$, constant
$\rho g z - \frac{1}{2}\rho\Omega^2(x^2+y^2) = p_a - p_0$ is constant

$z = \dfrac{\Omega^2}{2g}(x^2+y^2) + \text{constant}$ (as last year)

Now we introduce the dynamic pressure p_D, which is the deviation from equilibrium from equilibrium pressure, i.e. write $p_D = p - p_e$
or $p = p_e + p_D$.

Then $\dfrac{D\underline{u}}{Dt} + 2\underline{\Omega}\wedge\underline{u} = -\dfrac{1}{\rho}\nabla p - \nabla G_c - \nabla G_g$
$= -\dfrac{1}{\rho}\nabla p_D - \cancel{\dfrac{1}{\rho}\nabla p_e} - \cancel{\nabla G_c} - \cancel{\nabla G_g}$

i.e. $\boxed{\dfrac{D\underline{u}}{Dt} + 2\underline{\Omega}\wedge\underline{u} = -\dfrac{1}{\rho}\nabla p_D}$ our governing equation

This is useful except when there is a free surface present (without further modification)

Thus we have $\boxed{\begin{array}{l}\dfrac{D\underline{u}}{Dt} + 2\underline{\Omega}\wedge\underline{u} = -\dfrac{1}{\rho}\nabla p_D \\ \nabla\cdot\underline{u} = 0\end{array}}$ 4 equations in 4 unknowns

$[p = p_e + p_D]$

$$\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{u}.\underline{\nabla})\,\underline{u}$$

→ relative to the rotating frame

For steady flow, $\frac{\partial}{\partial t} = 0$ and slow flow, $|u| \ll 1$, so $|u|^2 \lll 1$,

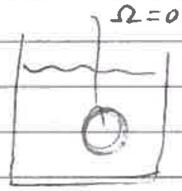
i.e. $|(\underline{u}.\underline{\nabla})\underline{u}| \sim |\underline{u}|^2 \lll 1$.

$$\Rightarrow \frac{D\underline{u}}{Dt} = \cancel{\frac{\partial \underline{u}}{\partial t}}^{\text{steady}} + \cancel{(\underline{u}.\underline{\nabla})\underline{u}}^{\text{slow}}$$

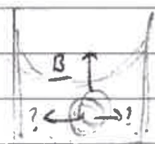
$$\& \quad \cancel{\frac{D\underline{u}}{Dt}}^{\text{steady \& slow}} + 2\underline{\Omega} \wedge \underline{u} = -\frac{1}{\rho}\nabla p_0 \quad \Rightarrow \text{geostrophic balance}$$

HW2 Archimedes principle:

$$-\int p_e\, \hat{n}\, dS$$
$$= -\int \nabla p\, dV$$
$$-\hat{z}\int_V g + \rho^2 \int \ldots \hat{z}$$



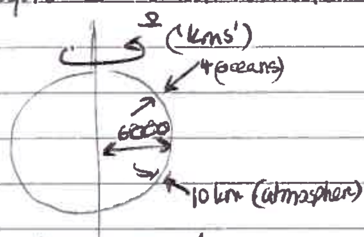
lower a body into a fluid
- subject to an upward force = weight of fluid displaced



What are the other forces & how large are they

Chapter 2: Shallow Water Equations.

21/01



$\underline{u}$: relative to a frame in solid body rotation

$$\frac{D\underline{u}}{Dt} + 2\underline{\Omega} \wedge \underline{u} = -\frac{1}{\rho}\underline{\nabla} p_0 + \underline{F}$$

Thin layer equations.

$$\begin{pmatrix} p = p_e + p_0 \\ p_e = p_0 - \rho g z + \frac{1}{2}\rho\, \Omega^2 r^2 \end{pmatrix}$$

$\rho \sim \rho_0 e^{-z/H_s}$

H_s scale heights

Incompressible. $M \ll 1$.

Planetary scale motions $L \sim 10{,}000\text{km}$

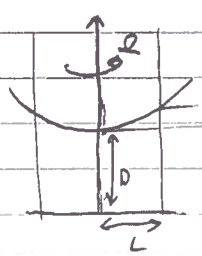
depths $D \sim 4\text{km}$.

$\delta = \frac{D}{L} \ll 1$.

Approximate our equations in limit $D/L \ll 1$.

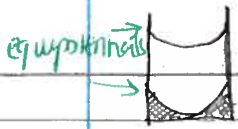
This year, look at the leading order behaviour (errors $\sim \left(\frac{D}{L}\right)^2$)

(Modern research: higher order theories - expand further in D/L.



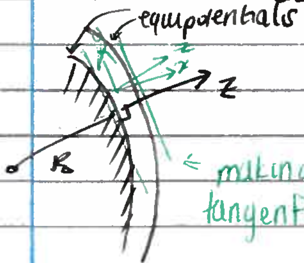
$$\frac{\Delta D}{D} \sim \frac{\Omega^2 L}{2g}$$

- limits how fast spin your container when interested in surface waves.



Or, we could build the bottom to have same paraboloid on surface
– only works at one rotation rate.

The earth's surface and the bottom of the ocean is an equipotential of pressure $\rho g z - \frac{1}{2}\Omega^2 r^2$
And Earth's rotation rate is fixed.
$\Rightarrow$ we measure displacements perpendicular to the gravitational equipotentials and then can treat p_e as a constant on the surface.



← making a local tangent plane approximation.

Really should use ϕ azimuth – longditude
θ latitude
z local vertical (in direction of $\hat{R}$).

It is sufficient in general to simply use cartesian coordinates.
y North, x East, z up.

(Error is of order $(\Delta\theta)^2$) because it is a linear approximation.

Shallow water equations

later gets eliminated

(1) $$\underset{U/T}{\frac{\partial u}{\partial t}} + \underset{U^2/L}{u\frac{\partial u}{\partial x}} + \underset{U^2/L}{v\frac{\partial u}{\partial y}} + \underset{WU/D}{w\frac{\partial u}{\partial z}} - \underset{fU}{2\Omega v} = \underset{P/\rho L}{-\frac{1}{\rho}\frac{\partial p}{\partial x}}$$

(2) $$\underset{U/T}{\frac{\partial v}{\partial t}} + \underset{U^2/L}{u\frac{\partial v}{\partial x}} + \underset{V^2/L}{v\frac{\partial v}{\partial y}} + \underset{WU/D}{w\frac{\partial v}{\partial z}} + \underset{fU}{2\Omega u} = \underset{P/\rho L}{-\frac{1}{\rho}\frac{\partial p}{\partial y}}$$

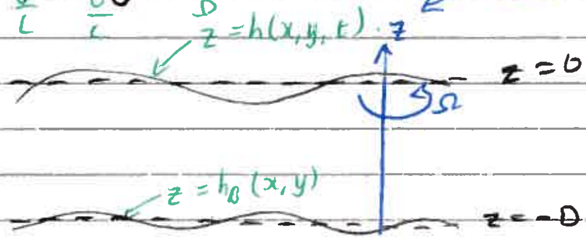
← typical scales of the terms in these equations

(3) $$\underset{W/T}{\frac{\partial w}{\partial t}} + \underset{UW/L}{u\frac{\partial w}{\partial x}} + \underset{UW/L}{v\frac{\partial w}{\partial y}} + \underset{W^2/D}{w\frac{\partial w}{\partial y}} = \underset{P/\rho D}{-\frac{1}{\rho}\frac{\partial p}{\partial z}}$$

(This later becomes $\frac{\partial p}{\partial z} = 0$).

(4) $$\underset{U/L}{\frac{\partial u}{\partial x}} + \underset{U/L}{\frac{\partial v}{\partial y}} + \underset{W/D}{\frac{\partial w}{\partial z}} = 0$$

Taken the rotation to be about the local vertical
↳ "TRADITIONAL" APPROXIMATION.



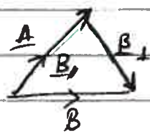
← flat equipotential surface of the ocean

← bottom

Let U be a typical magnitude for a horizontal velocity and W be a typical vertical velocity $\frac{W}{U} \sim \frac{D}{L} \ll 1$.

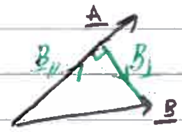
Now
$$\underline{A} \wedge \underline{B} = \underline{A} \wedge \underline{B}_{\parallel} + \underline{A} \wedge \underline{B}_{\perp}$$
$\underline{A} \wedge \underline{B}_{\parallel} \to 0$

$\underline{B}_{\parallel} + \underline{B}_{\perp} = \underline{B}$ and $\underline{B}_{\parallel}$ is parallel to $\underline{A}$



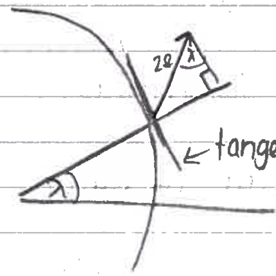
$$\Rightarrow 2\underline{\Omega} \wedge \underline{u} = f\hat{\underline{z}} \wedge \underline{u}.$$

When we form the cross product of two vectors, only the perpendicular component of one vector contributes. ie. $\underline{A} \wedge \underline{B} = \underline{A} \wedge \underline{B}_\perp$



So since to leading order (in D/L) $\underline{u}$ is horizontal $\quad \left(\frac{W}{U} \sim \frac{D}{L} \ll 1\right)$

$2\underline{\Omega} \wedge \underline{u} = f\hat{\underline{z}} \wedge \underline{u}$ where f is the local vertical component of $2\underline{\Omega}$.



$f = 2\Omega \sin\lambda$ $\quad$ (f - Foucault's pendulum) $\sim$ coriolis parameter.

24/01 $\quad 2\underline{\Omega} = (2\Omega\sin\lambda)\hat{\underline{z}} = f\hat{\underline{z}}$

$\hat{\underline{z}} \wedge \underline{u} = -v\hat{i} + u\hat{j} = 0\hat{\underline{z}} \qquad \underline{u} = u\hat{i} + v\hat{j} + w\hat{\underline{z}}$

Let the motion have a typical horizontal scale L
horizontal scale D
time T
horizontal vel. scale U
vertical vel. scale W
pressure variation P

In the shallow water limit, we take $\delta = D/L \ll 1$.

We treat equation (4) first: $\dfrac{\partial w/\partial z}{\max\left\{\frac{\partial v}{\partial y}, \partial u/\partial x\right\}} \sim \dfrac{W}{D} \times \dfrac{L}{U} = \dfrac{WU}{\delta}$

For this ratio to remain finite as $\delta \to 0$, we must have $\dfrac{W}{U} \leq O(\delta)$

ie. $\dfrac{W}{U} \to 0$ at least as fast as δ (it could go faster, but not slower).

ie $\boxed{W \leq \delta U}$ ie. $\boxed{W \ll U}$

By a similar argument, in (1) & (2), $\dfrac{P}{\rho L} = \max\left[\dfrac{U}{T}, \dfrac{U^2}{L}, \dfrac{WU}{D}, fU\right]$

$P = \rho U \left[\dfrac{L}{T}, U, fL\right]_{max}$.

N.B: $\frac{W}{D} = \frac{W}{\delta}$
$W \leq \delta U$
$\frac{W}{\delta} \leq U$
$\Rightarrow \frac{WU}{D}$ cannot exceed $\frac{U^2}{L}$ by (4)

Now look at (3). (Our aim is to show $\dfrac{\partial p}{\partial z} = 0$)

$$\frac{LHS}{RHS} = \frac{\left|\rho \frac{Dw}{Dt}\right|}{\left|\frac{\partial p}{\partial z}\right|} \leq \frac{\rho\left[\frac{W}{T}, \frac{UW}{L}\right]_{max}}{P/D} = \frac{\rho\left[\frac{W}{T}, \frac{UW}{L}\right]_{max}}{\frac{\rho U}{D}\left[\frac{L}{T}, U, fL\right]_{max}} = \frac{\frac{\rho WD}{\rho UL}\left[\frac{1}{T}, \frac{U}{L}\right]_{max}}{\left[\frac{1}{T}, \frac{U}{L}, f\right]_{max}}$$

$$\leq \frac{\delta^2\left[\frac{1}{T}, \frac{U}{L}\right]_{max}}{\left[\frac{1}{T}, \frac{U}{L}, f\right]_{max}} \leq \delta^2 \text{ always} \qquad \delta \ll 1, \; \delta^2 \lll 1.$$

$$\left(\partial_{\tilde{z}} \tilde{z} = O\delta^2\right)$$

Thus $\boxed{(3) \text{ becomes } \frac{\partial p}{\partial z} = 0}$ ie. the pressure is constant through the layer.

Examples: Lubrication theory, boundary layers, tea leaves.

In our case it is even better, because if the container is rotating sufficiently fast enough ie. f is sufficiently large, then the ratio can be small even where δ is not small. This will be true if $\frac{U}{fL} \ll 1$.

(don't even need to have a shallow layer)

This is the Rossby number, $\varepsilon = U/fL$

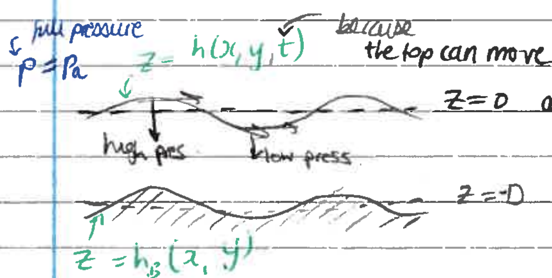
Ratio for $\varepsilon < 1$ is $\varepsilon\delta^2$. This is small for $\delta \sim 1$ if $\varepsilon \ll 1$
ie. rapidly rotating (ie small Rossby number) deep flows behave like shallow flows
ie. they have depth-independant pressure.
$\Rightarrow$ we can do an experiment in a deep tank with Rossby number small & this is the same as a shallow layer on earth. (not true if not rotating, only for rapidly rotating)
ie The pressure becomes depth indepent for flows rotating rapidly enough.

In (1) and (2), the pressure forcing (RHS) is independant of z $\Rightarrow$ the acceleration is independant of z (LHS). Thus if the initial conditions are independant of z for (u,v) [w can be dependant of z], (u,v) will remain independant of z $\forall$ time.
Thus we can look for solutions for (u,v) in the forms $u = u(x,y,t)$
$v = v(x,y,t)$

ie. $w\frac{\partial u}{\partial z}$ and $w\frac{\partial v}{\partial z}$ in (1) and (2) disappear.

$\Rightarrow$ Starting to look like 2D fluid dynamics. No w in (1), (2) or (3)!

But we need an equation for $w(x,y,z,t)$. This has to come from equation (4)



slope of the slope gives us the pressure gradient.

remember that p in (3) is the dynamic pressure, p_D

Integrating (3) gives $p_D = p_D(x,y,t)$ so the total pressure is $p = p_e + p_D$
$= p_D(x,y) - \rho g z$
(takes $p_e = p_h = p_0 - \rho g z$) hydrostatic

Thus on $z = h(x,y,t)$, $p_{atm} = p_D(x,y,t) - \rho g h(x,y,t)$ so $\underset{\sim}{\nabla} p_D = \rho g \underset{\sim}{\nabla} h$

Could be asked to derive these (with guidance).

$\therefore$ RHS of (1) & (2) become $-g\frac{\partial h}{\partial x}$ and $-g\frac{\partial h}{\partial y}$

ie. We have
$$\boxed{\begin{aligned}\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv &= -g\frac{\partial h}{\partial x} \quad (1)\\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + fu &= -g\frac{\partial h}{\partial y} \quad (2)\end{aligned}}$$

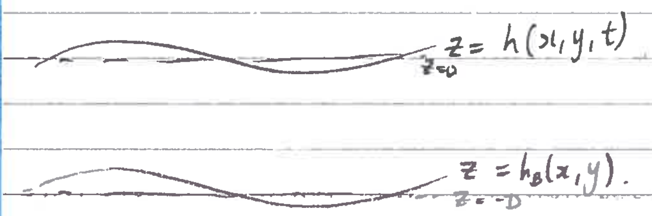
SHALLOW WATER MOMENTUM EQUATIONS

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (4)$$

Now have unknowns u, v, w, h

equations ①, ②, ④.

p : hydrostatic.



$p_r = p_o - \rho g z + p_D$

$p_{atm} = p_o - \rho g h + p_D$

$p_r - p_{atm} = \rho g(h-z) \Rightarrow \underline{\underline{p_{TOT}}} = p_{atm} + \underbrace{\rho g(h-z)}_{\text{weight of water above you}}$

plays the role of p_o

If you know h, you know p, so pressure will no longer be mentioned.

(need to eliminate w like how we eliminated p).

Now (4) says $\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$.

But RHS is independant of z. Thus integrate from $z = h_B$ to $z = h$ to get

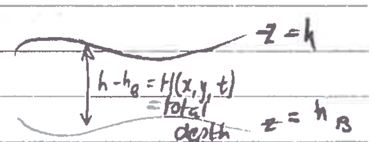
$$w(z=h) - w(z=h_B) = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(h - h_B)$$

For a particle on the surface, for all time, $z = h(x,y,t)$ $\forall t$.
Following the particle, we have that $\frac{Dz}{Dt} = \frac{Dh}{Dt}$ on surface $z = h(x,y,t)$

Similarly, for particle on the bottom, $w = \frac{Dh_B}{Dt}$ on $z = h_B(x,y)$.

From now on, it is sufficient to take $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ since h, h_B do not contain z.

Then we have $\frac{D}{Dt}(h - h_B) = \frac{Dh}{Dt} - \frac{Dh_B}{Dt} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(h - h_B)$



ie. write this as a total depth, $H(x,y,t) = h - h_B$

Then $\frac{1}{H}\left(\frac{DH}{Dt}\right) = -\underline{\nabla} \cdot \underline{u}$

where, once again, we only need the 2D operator $\underset{\sim}{\overset{\text{grad}}{\nabla}} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}$
and use

[finished]

Summary:

$$\boxed{\begin{aligned} \frac{Du}{Dt} - fv &= -g\frac{\partial h}{\partial x} \\ \frac{Dv}{Dt} + fu &= -g\frac{\partial h}{\partial y} \\ \frac{DH}{Dt} + H\nabla\cdot\underline{u} &= 0 \end{aligned}}$$

ROTATING SHALLOW WATER EQUATIONS (SWE).

"h plays the role of pressure"
"H plays the role of density".

$u(x,y,t)$, $v(x,y,t)$, $h(x,y,t)$

Note this is a closed system for u, v, h since $H = h - h_B$ & h_B is given.
This is a 2D system in x, y (and time) whereas the original system was 3D.
- many similarities with 2D fluid dynamics — exact analogy with compressible 2D

NOTE: $\frac{DH}{Dt} = \frac{\partial H}{\partial t} + (\underline{u}\cdot\nabla)H$ so our conservation of mass can be written as

$$\frac{\partial h}{\partial t} + \nabla\cdot(H\underline{u}) = 0 \quad \text{or} \quad \frac{\partial h}{\partial t} + \nabla\cdot((h-h_B)\underline{u}) = 0$$

Where's w gone? we have $\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$.

Integrate from $z = h_B$ to z (arbitrary z, not necessarily to surface)

$$w(z) - w(h_B) = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(z - h_B) \qquad (**)$$

$$\text{i.e. } w = \frac{Dh_B}{Dt} - (\nabla\cdot\underline{u})(z - h_B)$$

$$w = (\underline{u}\cdot\nabla)h_B - (\nabla\cdot\underline{u})(z - h_B) \quad \Rightarrow \text{a linear function of } z$$

For these equations, we will look at 1. General properties
2. Waves

General properties of the SWE

From $(**)$ $\frac{D}{Dt}(z - h_B) = \frac{Dz}{Dt} - \frac{Dh_B}{Dt} = w - \frac{Dh_B}{Dt} = -(\nabla\cdot\underline{u})(z - h_B)$

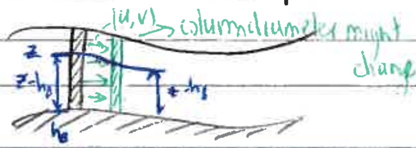
But $\frac{D}{Dt}(h - h_B) = -(\nabla\cdot\underline{u})(h - h_B)$

$$\frac{1}{z - h_B}\frac{D}{Dt}(z - h_B) - \frac{1}{h - h_B}\frac{D}{Dt}(h - h_B) = 0.$$

$$(h - h_B)\frac{D}{Dt}(z - h_B) - (z - h_B)\frac{D}{Dt}(h - h_B) = 0$$

$$\text{i.e. } \frac{D}{Dt}\left(\frac{z - h_B}{h - h_B}\right) = 0$$

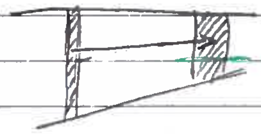
Comment: 1. Vertical fluid columns remain vertical throughout the motion since u and v are functions of x and y alone.



2. But w is non zero so height can change. But the change is trivial because following a particle, $\frac{z - h_B}{h - h_B}$ remains constant

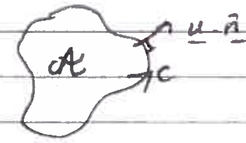
ie. a particle preserves its relative height in the column ie. $\frac{1}{3}$ way up stays $\frac{1}{3}$ up

3. Equation $\frac{DH}{Dt} + H\nabla.\underline{u} = 0$ is simply conservation of volume.



To conserve volume a column which shortens must fatten.

Proof: Take a cross-section of the column.



Let the instantaneous area of $\mathcal{A}$ be A. The rate of increase of area is $\frac{dA}{dt} = \oint_C (\underline{u}.\hat{n})\, ds$ ⇒ the area is changing as a function of time.

$= \int_{\mathcal{A}} (\nabla.\underline{u})\, dA \simeq (\nabla.\underline{u})A$ for sufficiently small $\mathcal{A}$ so $\nabla.\underline{u}$ is constant.

Thus $\nabla.\underline{u} = \frac{1}{A}\frac{DA}{Dt}$ ⇒ factor rate of increase of area of a small region.

25/01. **SWE** 3D Euler + Rotation

$\downarrow \delta = D/L \ll 1$

rSWE

→ (approx) Conservation relations

→ Linearised solution ← we will look at these later

→ [Quasigeostrophic limit. $\omega \ll f$.]

Conservation relations

1. Vertical columns remain vertical
2. Particle maintains rel. height in column
3. For a column of cross-section A, with are $A \ll 1$, $\nabla.\underline{u} \simeq \frac{1}{A}\frac{dA}{dt}$

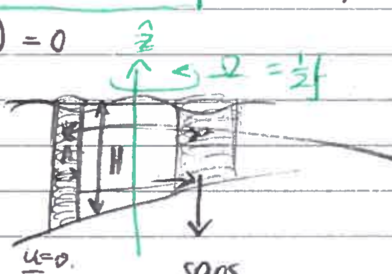
(In fact $\nabla.\underline{u} = \lim_{A\to 0}\frac{1}{A}\frac{dA}{dt}$) CONSERVATION OF VOLUME OR INCOMPRESSIBILITY $\frac{DH}{Dt} = \frac{dH}{dt}$ in this case.

But we have, from the SWE, $\boxed{\frac{DH}{Dt} + H\nabla.\underline{u} = 0}$ ie. $\frac{DH}{Dt} + \frac{H}{A}\frac{DA}{Dt} = 0$

ie. $A\frac{DH}{Dt} + H\frac{DA}{Dt} = 0$ ie. $\frac{D}{Dt}(AH) = 0$

ie. the column preserves AH — volume

4. Important



angular momentum relative to an vertical frame is non zero even when $\underline{u} = 0$ (by conservation of ang. mom)

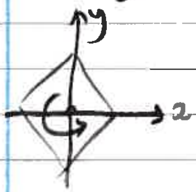
⇒ it has -ve relative velocity $\zeta = \hat{z}$

$\underline{\omega} = \zeta\hat{z}$.

movement generates relative vorticity from 0

except constant depth. Never have a vel. pot ϕ. ⇒ rel. vorticity is everywhere in rotating flows

Vorticity = 2x angular velocity of a fluid element about its centre of mass



$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2 \times \text{rate of rotation}.$$ proved in 2301

Thin column A; area $A \ll 1$

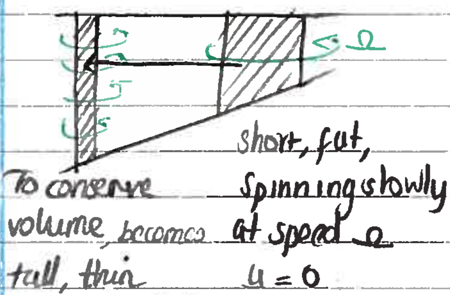
The total angular momentum of this column is $\frac{1}{2}\zeta_{absolute} A$.

This is conserved.

We have just shown that the volume HA is conserved

Dividing shows that the ratio $\boxed{q = \frac{\zeta_{absolute}}{H}}$ is conserved.

The quantity q is called the potential vorticity. (PV).



1. To conserve vol, becomes tall & thin
2. To conserve ang. mom, must spin faster.
(BALLERINA EFFECT).

Quantitative: eg double the depth, the double the absolute vorticity.

But parallel angular speeds/velocities add so $\frac{1}{2}\zeta_{abs} = \Omega + \frac{1}{2}\zeta_{rel}$.

ie. $\zeta_{abs} = \zeta + f$.

Hence $\boxed{q = \frac{\zeta + f}{H}}$ SHALLOW WATER PV.

HW: Get this from the SWE directly → it contains all the physics of the SWE

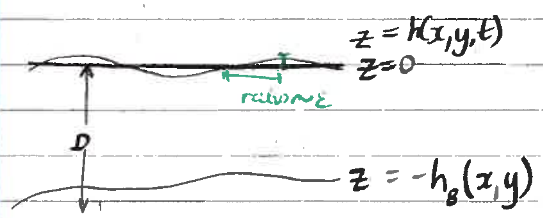
→ This is equivalent to the SWE with $q = \frac{\zeta + f}{H}$

$\boxed{\frac{Dq}{Dt} = 0}$ $\left[\frac{Du}{Dt} - fv = -g\frac{\partial h}{\partial x}\ ^{(1)}, \quad \frac{Dv}{Dt} + fu = -g\frac{\partial h}{\partial y}\ ^{(2)}\right]$ & $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

↳ Look at $\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$ and we get an equation for ζ, $\frac{D\zeta}{Dt}$.

Then eliminate $\nabla \cdot \mathbf{u} = -\frac{1}{H}\frac{DH}{Dt}$

Linearised Shallow water equations



Consider small (of order $\varepsilon \ll 1$) perturbations from a state of rest.

ie. we take the surface slopes $\frac{\partial h}{\partial x} \sim \varepsilon$

(really small waves)

ie $1 : \varepsilon \; : \varepsilon \; : 1 \; : 1$ take lim as $\varepsilon \to 0$

$$\underset{\varepsilon}{\frac{\partial u}{\partial t}} + \underset{\varepsilon^2}{u\frac{\partial u}{\partial x}} + \underset{\varepsilon^2}{v\frac{\partial u}{\partial y}} - \underset{\varepsilon}{fv} = -\underset{\varepsilon}{g\frac{\partial h}{\partial x}}$$ Then $u \sim \varepsilon$

Thus in the limit as $\varepsilon \to 0$, $\boxed{\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x}}$ (1)

Similarly, $\boxed{\frac{\partial v}{\partial t} + fu = -g\frac{\partial h}{\partial y}}$ (2) (linear).

The final equation is $\frac{\partial h}{\partial t} + \nabla.(H\underline{u}) = 0$

But $H = h - h_B$ $\quad$ $H \to -h_B = H_0(x,y)$. The undisturbed depth.

$\varepsilon \quad 1$ $\quad \Rightarrow \frac{h}{|h_B|} \sim \varepsilon$

So $\frac{\partial h}{\partial t} + \nabla.(H\underline{u}) = 0$ becomes $\boxed{\frac{\partial h}{\partial t} + \nabla.(H_0\underline{u}) = 0}$ (linear) (3)

$H_0(x,y)$.

(1)(2)(3) Linearised SWE.

First: Express $\underline{u}$ in terms of h, from (1), (2) to substitute in (3) so that we have an equation in h alone (a single equation for a single unknown)

In vector $\frac{\partial \underline{u}}{\partial t} + f\hat{\underline{z}} \wedge \underline{u} = -g\nabla h$ (4)

$$\frac{\partial (1)}{\partial t}: \quad \frac{\partial^2 \underline{u}}{\partial t^2} + f\hat{\underline{z}} \wedge \frac{\partial \underline{u}}{\partial t} = -g\nabla\frac{\partial h}{\partial t}$$

$$f\hat{\underline{z}} \wedge (4): \quad f\hat{\underline{z}} \wedge \frac{\partial \underline{u}}{\partial t} - f^2\underline{u} = -gf\hat{\underline{z}} \wedge \nabla h \qquad \hat{\underline{z}} \wedge (\hat{\underline{z}} \wedge \underline{u}) = -\underline{u}.$$

Subtracting: $\left(\frac{\partial^2}{\partial t^2} + f^2\right)\underline{u} = -g\nabla\frac{\partial h}{\partial t} + gf\hat{\underline{z}} \wedge \nabla h$ (5)

$\underline{u}$ in terms of h, from momentum.

Now split our consideration. First, take the ocean bottom to be flat and horizontal. ie. H_0 is a constant.
Later, return and allow $H_0 = H_0(x,y)$.

If H_0 is constant then $\frac{\partial h}{dt} + \nabla.(H_0(x,y)\underline{u}) = 0$

becomes $\frac{\partial h}{\partial t} + H_0\nabla.\underline{u} = 0$. (7).

$$\nabla.(5): \quad \left(\frac{\partial^2}{\partial t^2} + f^2\right)\nabla.\underline{u} = -g\nabla^2\frac{\partial h}{\partial t} + 0.$$

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)(7): \quad \left(\frac{\partial^2}{\partial^2 t} + f^2\right)\frac{\partial h}{\partial t} - gH_0\nabla^2\frac{\partial h}{\partial t} = 0$$

KLEIN GORDON EQUATION

$c^2 = gH_0$, integrate wrt: $\boxed{\left(\frac{\partial^2}{\partial t^2} + f^2\right)h = c^2\nabla^2 h}$

$\hookrightarrow$ If $f = 0$, $h_{tt} = c^2\nabla^2 h$ as expected.

ie. f^2 term as the rotation

31/01

rSWE

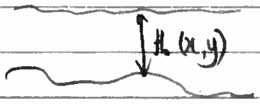
Nonlinear conservation properties

$\boxed{\frac{Dq}{Dt} = 0, \quad q = \frac{\zeta + f}{H}}$

linearise (waves) → lrSWE

lrSWE → variable background depth $H_0(x,y)$

lrSWE → flat bottomed case, constant H_0



H_0 undisturbed depth

- $(\partial_{tt} + f^2)\underline{u} = -g\underline{\nabla}\frac{\partial h}{\partial t} + fg\hat{\underline{z}} \wedge \underline{\nabla} h$ (whether flat or not)
- Flat bottom: $\frac{\partial h}{\partial t} + H_0 \underline{\nabla} \cdot \underline{u} = 0$

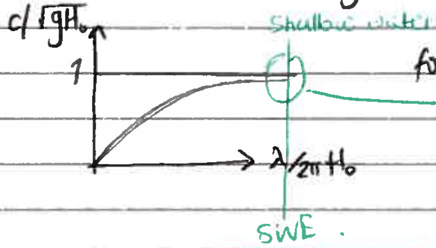
$H_0 = cst$ $\quad (\partial_{tt} + f^2) h_t + H_0 \nabla \cdot [(\partial_{tt} + f^2)\underline{u}] = 0$

ie. $(\partial_{tt} + f^2) h_t - g H_0 \nabla^2 h_t = 0$

gH_0 is a constant, dimensions $(\text{speed})^2$

ie. write $c^2 = g_0 H$, ie. $c = \sqrt{gH_0}$ where c is the longwave speed

↓ move fastest.



for water waves in general

we are looking at SHALLOW (lSWE)

so $\lambda/H_0 \to$ as $\varepsilon = D/L \sim H_0/\lambda \to 0$

In fact, this is $\boxed{h_{tt} + f^2 h = c^2 \nabla^2 h}$ KLEIN GORDON EQUATION.

having integrated wrt t.

If there is no rotation, $f = 0$, $h_{tt} = c^2\nabla^2 h$ (the usual 2D wave equation).

Whats the effect of f?

General wave jargon: Notice our equation (KG) is linear with constant coefficients

$\Rightarrow$ it has exponential solutions.

$\Rightarrow$ we can look for solutions of the form

$h(x,y,t) = \mathrm{Re}\{A e^{i(kx + ly - \omega t)}\}$

where A, k, l, ω are constants.

Here A = Amplitude of wave (arbitrary for a linear problem)

$\theta = kx + ly - \omega t$ (or $\theta = kx + ly + mz - \omega t$ in 3D)

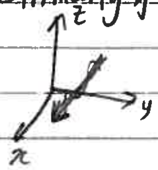
= the phase of the wave. (Phase = old name for argument).

ie. the wave repeats itself every time θ increases by 2π.

eg. if t increases by $2\pi/\omega$ with x,y,z fixed, θ increases by 2π.

& wave is unchanged. ie. (temporal) period is $T = 2\pi/\omega$.

Similarly if y,z,t are fixed, ie at a fixed time, we look in a yz plane



The wave has (spatial) period $\frac{2\pi}{k}$, ie. k is the number of waves in a distance 2π in the x direction

ie. k is the x-wavenumber

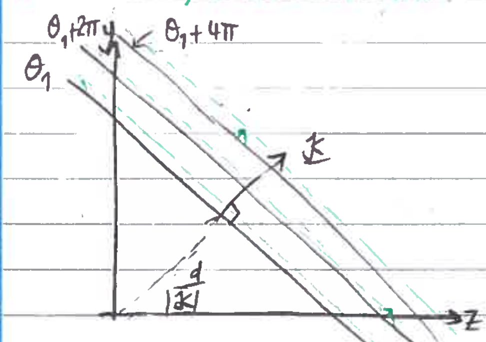
Similarly l is the y-wavenumber

m is the z wavenumber.

Notice $\theta = kx + ly + mz - \omega t = \underline{k}\cdot\underline{r} - \omega t$
where $\underline{k} = k\hat{\underline{x}} + l\hat{\underline{y}} + m\hat{\underline{z}}$, $\quad \underline{r} = x\hat{\underline{x}} + y\hat{\underline{y}} + z\hat{\underline{z}}$. $\qquad \underline{k} = (k, l, m)$
$\underline{r} = (x, y, z)$

At any given instant, the surfaces of constant phase are the surfaces θ = constant
ie. surfaces $\underline{k}\cdot\underline{r} - \omega t$ = constant
But t is fixed, so $\underline{k}\cdot\underline{r} = d$ (say) for some constant d.
ie. the surface of constant phase is a plane, with normal $\underline{k}$ at a distance $\frac{d}{|\underline{k}|}$ from O

Thus these waves are often called PLANE waves (because their lines of constant phase are planes)
(but in 2D, these are lines with normal $\underline{k}$ at distance $\frac{d}{|\underline{k}|}$ from O)



If we chose $e^{i\theta} = \pm 1$, $\theta = 0, 2\pi, 4\pi$ etc, this would be a crest (have same normal)
ie get a set of crests at any given time

line of constant phase (eg a crest)

At general time t, the plane with phase θ_0 satisfies $\underline{k}\cdot\underline{r} - \omega t = \theta_0$
ie. $\underline{k}\cdot\underline{r} = \underbrace{\theta_0 + \omega t}_{d(t)}$ $\Rightarrow$ Distance of this phase surface from the origin is $\frac{d}{|\underline{k}|} = \frac{\theta_0}{|\underline{k}|} + \frac{\omega t}{|\underline{k}|}$ ie. propogates at constant speed $\frac{\omega}{|\underline{k}|}$ away from the origin.
maintains the normal $\underline{k}$ ie green line.

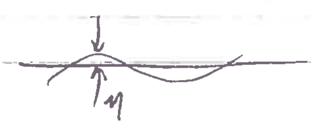
ie in general, the wave propogate in the direction $\underline{k}$ ie perpendicular to the phase lines at speed $\boxed{\frac{\omega}{|\underline{k}|} = \text{PHASE SPEED } c_p}$ $\rightarrow$ in 1D, $c_p = \frac{\omega}{k}$
in 2D $c_p = \frac{\omega}{\sqrt{k^2+l^2}}$
in 3D $c_p = \frac{\omega}{\sqrt{k^2+l^2+m^2}}$

The wavelength λ is the b/w distance between 2 crests at a given time; $\therefore$ one crest has $d = \theta_0/|\underline{k}|$
the next has $d = (\theta_0 + 2\pi)/|\underline{k}|$
so $\boxed{\lambda = \frac{2\pi}{k}}$

The $\boxed{c_p = \frac{\omega}{|\underline{k}|} = \frac{\lambda}{2\pi}\frac{2\pi}{T} = \frac{\lambda}{T}}$ since in time T, a crest moves distance λ to occupy the position previously occupied by the crest in front of it.

Return to the Klein Gordon. Take $h = \eta$ (just a notational change).

Then $\eta_{tt} + f^2\eta = c^2\nabla^2\eta$. $\qquad$ (1).



Look for plane wave solutions, ie. try $\eta = Re\{Ae^{i(kx+ly-\omega t)}\}$

For this to be a solution of (1), it is sufficient for $\eta = Ae^{i(kx+ly-\omega t)}$ to be a solution

Then $\frac{\partial \eta}{\partial x} = ik\eta \qquad \eta_{xx} = -k^2\eta \qquad \eta_{tt} = -\omega^2\eta$

So K-G becomes $(f^2-\omega^2)\eta = c^2(-k^2-l^2)\eta$

For non trivial solutions $(\eta \neq 0)$ $\quad f^2 - \omega^2 = -c^2(k^2+l^2)$.

ie. $\omega^2 = c^2(k^2+l^2) + f^2 = c^2\kappa^2 + f^2 \qquad \kappa = |\underline{\kappa}| = \sqrt{k^2+l^2}$

A relation between wavenumber (k,l) and frequency ω, is a dispersion relation

In absense of rotation $\omega^2 = c^2\kappa^2$ so $c_p = \frac{\omega}{\kappa} = \pm c$

$\Rightarrow$ All waves travel at speed c - non dispersive

With rotation, $c_p = \frac{\omega}{\kappa} = \frac{\pm\sqrt{c^2\kappa^2+f^2}}{\kappa} = \pm c\sqrt{1+\frac{f^2}{\kappa^2c^2}}$

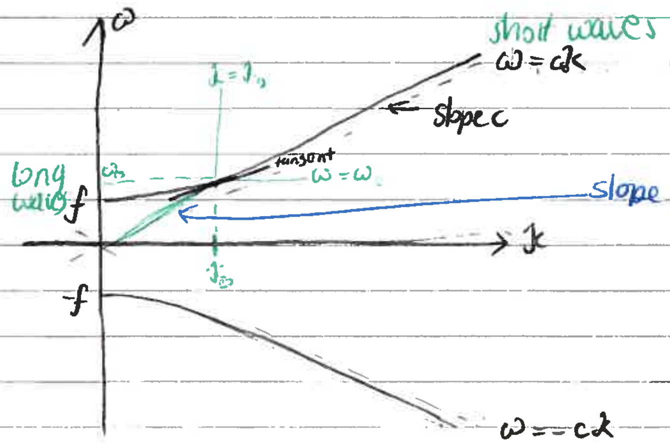
so $|c_p| > c$. Thus rotation increases wave speed and they depend on κ, So waves disperse (different waves travel at different speeds)

$\Rightarrow$ long waves travel fastest. $\omega \to \pm f \quad c_p = \frac{\omega}{\kappa} \Rightarrow \frac{\pm f}{\kappa}$ as $\kappa \to 0$

– rotation stiffens the surface $\Rightarrow$ waves travel faster

The waves are ISOTROPIC, ie they have the same speed in any direction (because they don't depend on k & l separately, but on $\kappa = \sqrt{k^2+l^2}$)

$\omega^2 = c^2\kappa^2 + f^2$ ie the dispersion relation:



slope of the chord is $\frac{\omega_0}{\kappa_0}$ = c_p

$\therefore$ $|c_p|$ is always greater than c

slope of the tangent gives c_g (group vel)

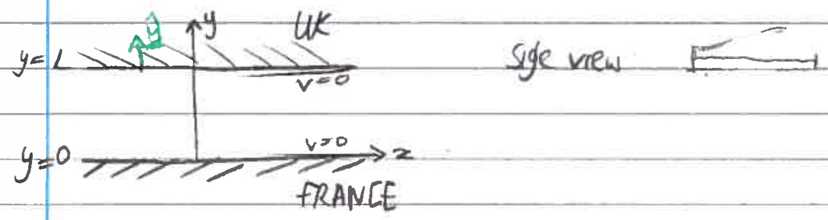
$|c_g| < c < |c_p|$

NB $(c_p c_g = c^2)$

These waves are called the POINCARÉ WAVES $\Rightarrow$ they are the rotation modified surface water waves important for tides

$h_{tt} + f^2 h = c^2 \nabla^2 h$ $\qquad \frac{2\pi}{f} \sim$ Coriolis period (1 day at the poles, infinite at the equator)

Example: Rotation effects on waves in a channel eg English channel



Governing equation : KG : $\eta_{tt} + f^2\eta = c^2\nabla^2\eta$

Boundary conditions : periodic in t
unbounded in x.
But $\underbrace{\underline{u}.\hat{n} = 0}_{\text{no normal flow}}$ at $y = 0, L$ $\forall x, t$ $\quad \hat{n} = \hat{y}$

So $\hat{y}.\underline{u} = 0$ on $y = 0, L$ $\forall x, t$.

Remember $(\partial_{tt} + f^2)\underline{u} = -g\nabla\eta_t + fg\hat{z}\wedge\nabla\eta$

Dotting with $\hat{y}$ gives $(\partial_{tt} + f^2)\underbrace{\underline{u}.\hat{y}}_{=0\ \forall t\ \forall x \text{ at } y=0,L} = -g\frac{\partial^2\eta}{\partial y\partial t} + fg\underbrace{\hat{y}.(\hat{z}\wedge\nabla\eta)}$

$(\hat{y}\wedge\hat{z}).\nabla\eta$ (we are allowed cyclic permutations)
$= \hat{x}.\nabla\eta$
$= \frac{\partial\eta}{\partial x}$

Thus $\frac{\partial^2\eta}{\partial y\partial t} = f\frac{\partial\eta}{\partial x} = 0 \quad y = 0, L \quad (\forall x, t)$

Look for plane wave (?) solutions. (Not quite.)

$\eta(x,y,t) = Re\{\bar{\eta}(y)\, e^{i(kx-\omega t)}\}$

because there are boundary conditions at finite y.

Then the BC's become $-i\omega\bar{\eta}' - ifk\bar{\eta} = 0$ at $y = 0, L$

Governing equation becomes $-\omega^2\bar{\eta} + f^2\bar{\eta} = c^2(-k^2\bar{\eta} + \bar{\eta}'')$

ie. $\bar{\eta}'' + \left[\frac{\omega^2 - f^2}{c^2} - k^2\right]\bar{\eta} = 0$

ie. $\bar{\eta}'' + \alpha^2\bar{\eta} = 0 \qquad \alpha^2 = \frac{\omega^2 - f^2}{c^2} - k^2$.

Sol^n is $\bar{\eta}(y) = A\sin\alpha y + B\cos\alpha y \qquad \alpha$ plays role of a wavenumber
$\omega^2 = (\alpha^2 + k^2)c^2 + f^2$

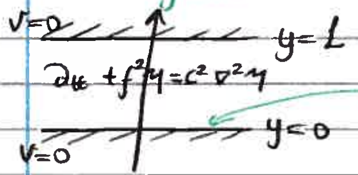
ie. these are POINCARÉ WAVES $\qquad \alpha \sim L$.

04/02.

Poincaré waves – rotation modified surface water waves

$\eta_{tt} + f^2\eta = c^2\nabla^2\eta$ $\qquad\qquad$ $\omega^2 = f^2 + c^2K^2$ ← (k^2+l^2)

eg tides



$f\eta_x - \eta_{yt} = 0$ $\qquad$ $\eta(x,y,t) = Re\{\bar\eta(y)e^{i(kx-\omega t)}\}$.

oblique BC (cannot separate variables)

$\bar\eta(y) = A\sin\alpha y + B\cos\alpha y$

At $y=0$, $A\alpha + \frac{kf}{\omega}B = 0$. $\Rightarrow \alpha A = -\frac{kf}{\omega}B$

At $y=L$, $A\alpha\cos\alpha L - B\alpha\sin\alpha L + \frac{kf}{\omega}(A\sin\alpha L + B\cos\alpha L) = 0$

$A\alpha^2\cos\alpha L - B\alpha^2\sin\alpha L + \frac{kf}{\omega}(\alpha A\sin\alpha L + \alpha B\cos\alpha L) = 0$

$\Rightarrow -\frac{kf}{\omega}B\alpha\cos\alpha L - B\alpha^2\sin\alpha L + \frac{kf}{\omega}\left(-\frac{kf}{\omega}B\sin\alpha L + \alpha B\cos\alpha L\right)$

$\Rightarrow B\sin\alpha L\left\{\alpha^2 + \frac{k^2f^2}{\omega^2}\right\} = 0$

(we use $(\eta_{tt}+f^2)\underline{u} = gf\hat{z}\wedge\nabla\eta - g\nabla\eta_t$)

NB: $B=0 \Rightarrow A=0 \Rightarrow$ no wave ie $\eta=0$. So $B\neq 0$.

Thus either $\sin\alpha L = 0$ or $\alpha^2 + \frac{k^2f^2}{\omega^2} =$

Case 1: $\sin\alpha L = 0 \Rightarrow \alpha L = n\pi$ $\quad n = 0, \pm1, \pm2, \ldots$ WLOG take $n = 0,1,2,3,\ldots$

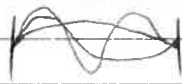
Thus $\alpha^2 = \frac{n^2\pi^2}{L^2}$ $\quad$ We had $\omega^2 = f^2 + c^2(k^2+\alpha^2)$

↳ poincaré waves for quantised wave no.

For unbounded Poincaré waves, we had $\omega^2 = f^2 + c^2(k^2+l^2)$

But not, we have the y-wave number, α, being quantised $\alpha = \frac{n\pi}{L}, n = 0,1,2,\ldots$

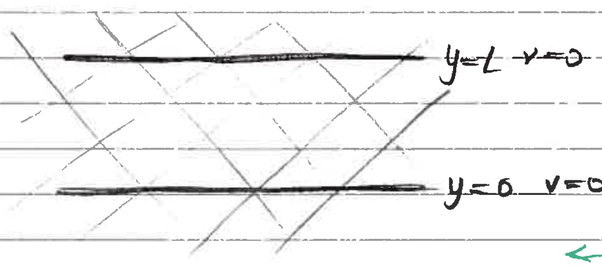
↳ (just like how wave numbers of a finite string are quantised.



Thus $\eta(x,y,t) = Re\{(A\sin\alpha L + B\cos\alpha L)e^{i(kx-\omega t)}\}$

$\cos\alpha y = \frac{1}{2}(e^{+i\alpha y} + e^{-i\alpha y})$ Thus our solution is a linear combination of $e^{i(kx+\alpha y-\omega t)}$ and $e^{i(kx-\alpha y-\omega t)}$.

ie. 2 PW with α y-wave number & phase lines sloping at $(y = \pm\frac{k}{\alpha}x)$, $\pm\frac{k}{\alpha}$



$kx + \alpha y - \omega t$ must be constant
$kx - \alpha y - \omega t$ must be constant

two waves (oblique) moving in opposite directions.

← periodic so would get v=0 here too

One of the solutions has $n = 0$. Then $\alpha = 0$. Then $\bar{\eta}(y) = A\sin\alpha y + B\cos\alpha y = B$.

So $\frac{\partial \bar{\eta}}{\partial y} = 0$ This says that $\bar{\eta}$ does not change moving down the channel.

Remember $\underbrace{(\partial_{tt} + f^2) v}_{\text{indep of } y} = -g\eta_{yt} + \underbrace{fg\eta_x}_{\text{indep of } y}$.

$n=0$ $v = 0$ at $y = 0 \Rightarrow v = 0$ everywhere. But $\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \Rightarrow u = 0$ ie. no wave

What has happened?

Case 2: $\alpha^2 + \frac{f^2 k^2}{\omega^2} = 0$ α could be imaginary

ie. $\alpha = \pm i f\frac{k}{\omega}$ Then could take $\bar{\eta}(y) = \tilde{A}\sinh\frac{fky}{\omega} + \tilde{B}\cosh\frac{fky}{\omega}$.

or (better), $\bar{\eta}(y) = \bar{A}\exp\left[\left(\frac{fk}{\omega}\right)y\right] + \bar{B}\exp\left[-\left(\frac{fk}{\omega}\right)y\right]$

First: Dispersion relation. $\omega^2 = f^2 + c^2(k^2 + \alpha^2)$
$= f^2 + c^2\left(k^2 - \frac{f^2}{\omega^2}k^2\right)$

$(\omega^2 - f^2) = c^2k^2\left(1 - \frac{f^2}{\omega^2}\right) = \frac{c^2k^2}{\omega^2}(\omega^2 - f^2)$

$(\omega \neq f)$ Contained in our solution (Pedlosky)

ie. $\omega^2 = c^2k^2$ ie $\omega = (\pm c)k$

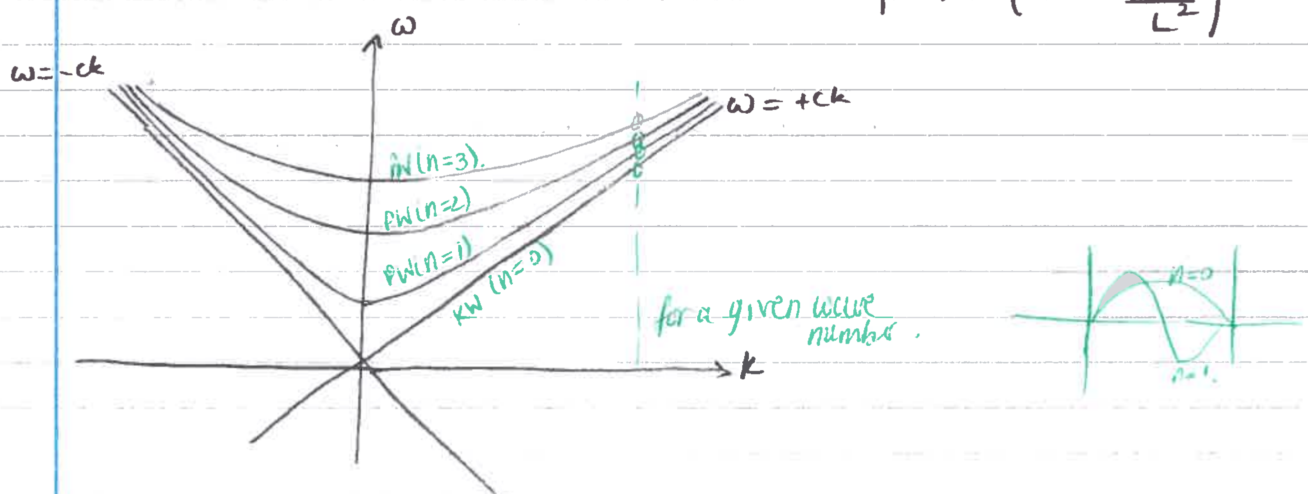
ie. these are non dispersive (ω is a linear fn of k) long waves unaffected in their speed by rotation

$C_p = \frac{\omega}{k} = \pm c$ (just as when $f = 0$). So rotation doesn't effect the speed of waves – KELVIN WAVE

(y structure – next time)

Lets plot the dispersion relation KWs: $\omega = (\pm c)k$

PWs: $\omega^2 = f^2 + c^2\left(k^2 + \frac{n^2\pi^2}{L^2}\right)$



7/02

We have $\bar{\eta}(y) = \bar{A} e^{kfy/\omega} + \bar{B} e^{-fky/\omega}$ (1) $\quad \eta(x,y,t) = Re\{\bar{\eta}(y) e^{i(kx-\omega t)}\}$ (2)

We can take $\omega = ck$ $\quad \bar{\eta}(y) = \bar{A} e^{y/a} + \bar{B} e^{-y/a}$ where $a = \frac{\omega}{kf} = \frac{ck}{kf} = \frac{c}{f}$

$[a] = \frac{[c]}{[f]} = \frac{LT^{-1}}{T^{-1}} = L \Rightarrow a$ is a length.

ROSSBY RADIUS.

ie. the motion has exponential decay over scales of order a.

Now remember $\left(\frac{\partial^2}{\partial t^2} + f^2\right) v = -g \frac{\partial^2 \eta}{\partial t \partial y} - fg \frac{\partial \eta}{\partial x}$.

Substitute (1) into here gives $\left(\frac{\partial^2}{\partial t^2} + f^2\right) v = -2\bar{A} f a k e^{fky/\omega} \sin(kx - \omega t)$

$v = 0$ $\forall t$ on $y = 0 \Rightarrow \bar{A} = 0$ No restriction on $\bar{B}$

$v = 0$ in this Kelvin wave. (Here we have proven v is zero - most texts will take this as an assumption.)

Thus there is no cross channel motion in a KW.
Hence $\eta(x,y,t) = \bar{B} e^{-fky/\omega} \cos(kx - \omega t)$

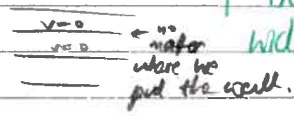
Two waves: $\omega = +ck$ $\quad \eta = \bar{B} e^{-y/a} \cos[k(x - ct)]$

$\omega = -ck$ $\quad \eta = \bar{B} e^{y/a} \cos[k(x + ct)]$

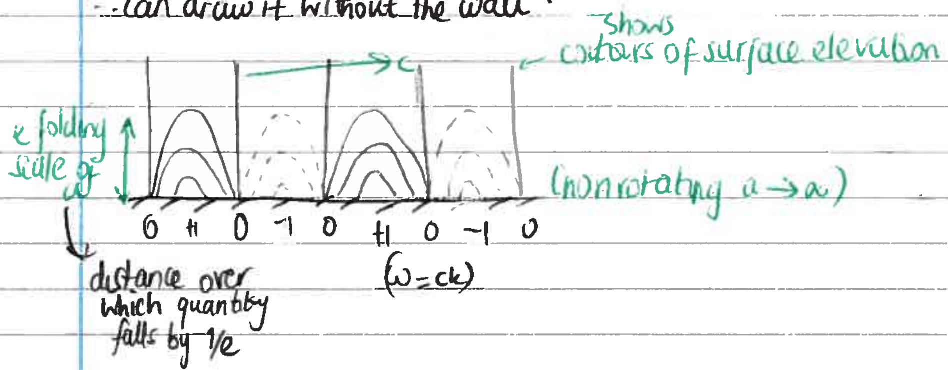
Both waves have $v = 0$

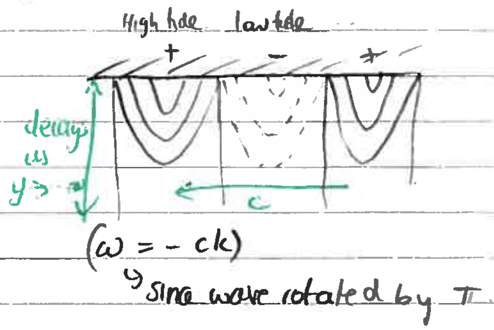
Both waves satisfy the BC regardless of the width of the channel

The width of the channel does not appear.



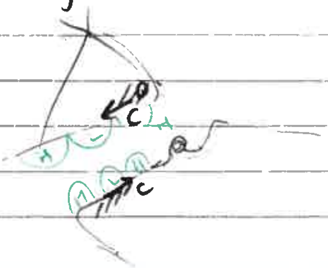
$\therefore$ can draw it without the wall:





ie. a KW always propagates with the supporting wall to its right.

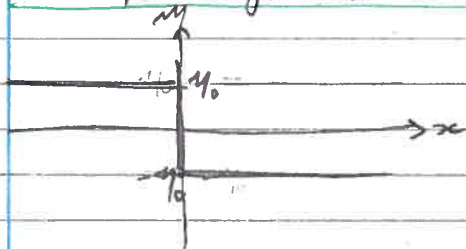
The only affect of rotation is to introduce a decay away from wall with scale $a = \frac{c}{f}$.



Recall: Slow $\rightarrow u^2$ neg $\Rightarrow$ linear.
Steady $\frac{\partial u}{\partial t}$ neg.
$2\underline{\Omega} \wedge \underline{u} = -\frac{1}{\rho}\nabla p$.

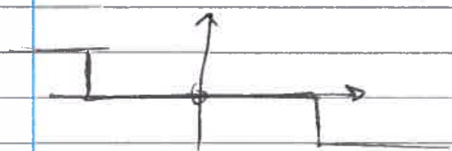
$f\hat{z} \wedge \underline{u} = -g\nabla\eta$. geostrophic.

Geostrophic Adjustment : Rossby - Gill



$$\frac{\partial^2 \eta}{\partial t^2} = c^2 \frac{\partial^2 \eta}{\partial x^2} \qquad \eta(x,0) = -\eta_0 \operatorname{sgn} x \quad ② \qquad \frac{\partial \eta}{\partial t}(x,0) = 0. \quad ③$$

RG adjustment, solve this with $f \neq 0$.



ie. $$\frac{\partial^2 \eta}{\partial t^2} + f^2 \eta = c^2 \frac{\partial^2 \eta}{\partial x^2} \quad ①$$

Fourier transforms. Let $\eta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\eta}(k,t) e^{ikx}\, dk$ ④

The Fourier Inverse Theorem then would given $\hat{\eta}(k,t) = \int_{-\infty}^{\infty} \eta(x,t) e^{-ikx}\, dx$

$\frac{\partial \eta}{\partial x} = \frac{1}{2\pi}\int_{-\infty}^{\infty} +ik\hat{\eta} e^{ikx}\, dx$ and so on.

Sub (4) into (1) : $\frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{d^2\hat{\eta}}{dt^2} + f^2\hat{\eta} + c^2k^2\hat{\eta}\right) e^{ikx}\, dk = 0$

Thus we seek a solution for $\hat{\eta}$ satisfying $\frac{d^2\hat{\eta}}{dt^2} + (f^2 + c^2k^2)\hat{\eta} = 0$

subject to (from (3)) $\frac{d\hat{\eta}}{dt} = 0$ at $t = 0$.

Thus $\hat{\eta} = A\sin\omega t + B\cos\omega t$ where $\omega^2 = f^2 + c^2k^2$. - the PW dispersion rel^n.

- our solution is a superposition of PWs.
(there is no boundary so there is no Kelvin waves in this problem).

Now $\frac{d\hat{\eta}}{dt} = \omega A\cos\omega t - \omega B\sin\omega t$. For this to vanish at $t=0$, $A=0$

Thus $\eta(x,t) = \frac{1}{2\pi}\int_{k=-\infty}^{\infty} B(k)\cos\omega t\, e^{ikx}\, dk$ - a superposition of Poincaré waves.

Here, $B(k)$ is determined by the initial conditions.

Here $B(k)$ is determined by the initial shape.

$$\eta(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} B(k)e^{ikx}\,dk = -\eta_0 \,\mathrm{sgn}\, x.$$

By the F.I.T., $B(k) = -\eta_0 \int_{-\infty}^{\infty} \mathrm{sgn}\, x\, e^{-ikx}\,dx$..

What analysis can we do on this?

$$\eta(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} B(k)e^{i(kx+\omega t)}\,dk + \frac{1}{4\pi}\int_{-\infty}^{\infty} B(k)e^{i(kx-\omega t)}\,dk \qquad \omega=\omega(k).$$

At large time ($t\to\infty$). $I = \int_{-\infty}^{\infty} A(k)\, e^{i\Phi(k)t}\,dk$

could integrate by parts - OK provided ϕ' doesn't vanish $\Rightarrow$ you get $\frac{1}{t}, \frac{1}{t^2}$ ....
If ϕ' vanishes. ie. the derivative of the phase vanishes
ie the phase is stationary $\Rightarrow$ use method of stationary phase.

But, (Rossby) PV is conserved in the rotating SWE.
so PV is conserved in the lrSWE.

lrSWE:
$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x} \qquad (5) \qquad \text{NB } h=\eta.$$
$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial h}{\partial y} \qquad (6)$$
$$\frac{\partial h}{\partial t} + D\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = 0 \qquad (7)$$

Introduce the vertical component of relative velocity, $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$.

$$\frac{\partial}{\partial x}(6) - \frac{\partial}{\partial y}(7) :- \quad \frac{\partial \zeta}{\partial t} + f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = 0 \qquad (8)$$

$$(8) - \frac{f}{D}(7) :- \quad \frac{\partial \zeta}{\partial t} - \frac{f}{D}\frac{\partial h}{\partial t} = 0.$$

ie. $\frac{\partial}{\partial t}\left(\zeta - f\frac{h}{D}\right) = 0.$ **

ie. $\left(\zeta - f\frac{h}{D}\right)_{t=t} = \left(\zeta - f\frac{h}{D}\right)_{t=0}.$

This is the linearised conservation of PV.

(SWE)

Now, $\frac{Dq}{Dt} = 0 \qquad q = \frac{\zeta + f}{H(x,y,t)}$.

$\frac{D}{Dt} = \frac{\partial}{\partial t} + \cancel{\underline{u} \cdot \nabla}$

$H = D + h(x,y,t) = D\left[1 + \frac{h}{D}\right]$

$\frac{1}{H} = \frac{1}{D}\left[1 + \frac{h}{D}\right]^{-1} = \frac{1}{D}\left(1 - \frac{h}{D} + \cdots\right)$

$= \frac{(\zeta + f)}{D}\left(1 - \frac{h}{D} + \cdots\right)$

$q = \frac{1}{D}(\zeta + f)\left(1 - \frac{h}{D} + \cdots\right) = \frac{1}{D}\left(\zeta + f - f\frac{h}{D} + \cdots\right)$

So $\frac{Dq}{Dt} = 0$ becomes $\frac{\partial}{\partial t}\left[\frac{1}{D}\left(\zeta + f - f\frac{h}{D} + \cdots\right)\right] = 0$

ie. $\frac{\partial}{\partial t}\left(\zeta - f\frac{h}{D}\right) = 0$ ** which is exactly what we had before.

We know PV at $t=0$ so we know PV for all time.

At $t = 0$ $u = 0$. so $\zeta = 0$ and $h = \eta = -\eta_0 \operatorname{sgn} x$

So $q = \frac{f\eta_0}{D} \operatorname{sgn} x$

So $\forall t$, $\zeta - f\frac{\eta}{D} = \frac{f\eta_0}{D} \operatorname{sgn} x \qquad (9)$

Assume the motion eventually becomes steady $\frac{\partial}{\partial t} \to 0$ (Rossby)

Then our momentum equations are $\cancel{\frac{\partial u}{\partial t}} - fv = -g\frac{\partial \eta}{\partial x}$

$\cancel{\frac{\partial v}{\partial t}} + fu = -g\frac{\partial \eta}{\partial y}$

ie motion becomes geostrophic.

Then $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{g}{f}\frac{\partial^2 \eta}{\partial x^2} + \frac{g}{f}\frac{\partial^2 \eta}{\partial y^2}$

$\zeta = \frac{g}{f}\nabla^2 \eta \leftarrow$ only true as $t \to \infty$ because of the assumption.

Then (9) becomes $\xrightarrow{\text{(no y dep)}}$ $\frac{g}{f}\frac{\partial^2 \eta}{\partial x^2} - \frac{f}{D}\eta = \frac{f}{D}\eta_0 \operatorname{sgn} x$

ie. $\frac{\partial^2 \eta}{\partial x^2} - \frac{f^2}{gD}\eta = \frac{f^2}{gD}\eta_0 \operatorname{sgn} x$

But $gD = c^2$ and $\frac{c}{f} = a$.

ie. $\frac{\partial^2 \eta}{\partial x^2} - \frac{1}{a^2}\eta = \frac{1}{a^2}\eta_0 \, \mathrm{sgn}\, x$

Notice that RHS is odd in x. So look for solution odd in x ie. $\eta(x) = -\eta(-x)$ $x<0$

Then $\eta(0) = 0$. and we need only to solve in $x \geq 0$

$$\eta_{xx} - \frac{1}{a^2}\eta = \frac{1}{a^2}\eta_0$$

$\eta_0 = 0$ at $x = 0$ $\qquad$ η bdd as $x \to \infty$.

Particular sol^n: $\eta_p = -\eta_0$

Comp. f^n: $\eta_c = Ae^{x/a} + Be^{-x/a}$

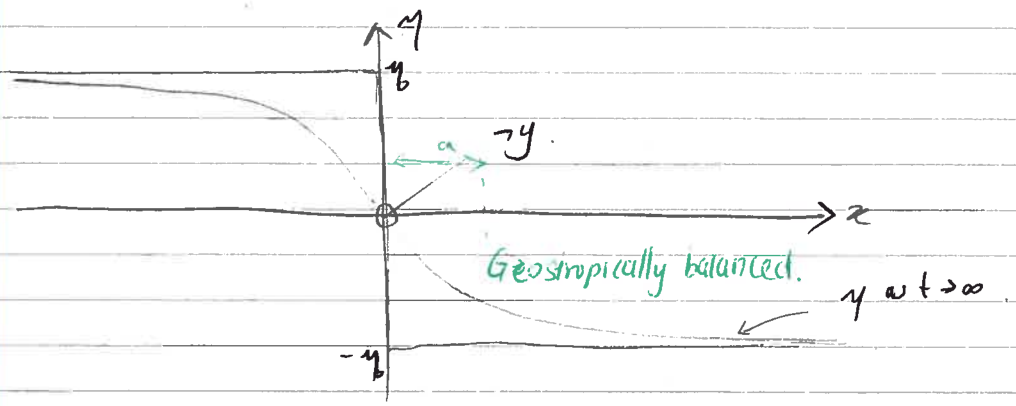
Bdd as $x \to \infty$, so $A \to 0$.

Gen Sol^n: $\eta = -\eta_0 + Be^{-x/a}$

But $\eta(0) = 0$ so $B = \eta_0$

ie $\eta = \begin{cases} -\eta_0 (1 - e^{-x/a}) & x > 0 \\ +\eta_0 (1 - e^{x/a}) & x < 0 \end{cases}$

$$\underline{\eta = -\eta_0 \, \mathrm{sgn}\left(1 - e^{-|x|/a}\right)}$$



Outside a region of width of order of the Rossby radius, nothing happens (no change).

$v = \frac{g}{f}\eta_x$. $\to$ flow is out of the page as $\eta_x < 0$ $\qquad$ NB. $\left(\frac{\partial}{\partial y} = 0 \not\Rightarrow v = 0\right)$

11/02 Linear : KG $\longrightarrow$ solve fully with FT.

$\searrow$ (Rossby) ① PV is :

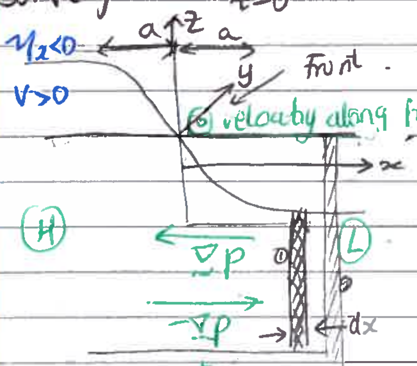
$$\zeta + \frac{\eta}{D} f \quad \text{is independant of } t.$$

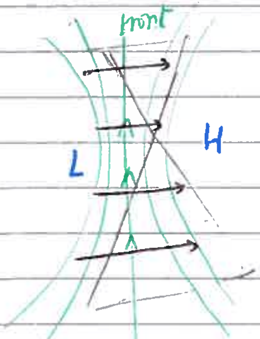
$$\left(\zeta + \frac{\eta}{D} f\right)_{t=\infty} = \left(\zeta + \frac{\eta}{D} f\right)_{t=0}$$

② + becomes steady $\Rightarrow$ $u = -\frac{g}{f}\eta_y$

$$v = +\frac{g}{f}\eta_x \qquad \zeta = fa^2\nabla^2\eta.$$

Solve for $\eta|_{t=0}$





The energetics of adjustment :

$\rightarrow$ looking at the column ①

- To leading order, the KE (per unit width in the y direction)

  $\hookrightarrow$ THIS IS ALWAYS TRUE IN A 2D PROBLEM.

  If a column of width δx in x-direction in the final state is $\frac{1}{2}(\rho D\delta x . 1)(u^2+v^2)$ $\qquad \frac{\partial\eta}{\partial y}=0$

  $\eta|_{t=\infty}$ $\qquad = \frac{1}{2}\rho D\,\delta x\left(\frac{g}{f}\frac{\partial\eta}{\partial x}\right)^2$ $\qquad \Rightarrow u=0$

Hence KE of the final state $= 2\int_0^\infty \frac{1}{2}\rho D\frac{g^2}{f^2}\left(\frac{\partial\eta}{\partial x}\right)^2 dx$

$$= \frac{\rho c^2}{f^2} g \int_0^\infty \frac{\eta_0^2}{a^2} e^{-2x/a}\, dx \qquad \eta = -\eta_0 \operatorname{sgn} x\,(1-e^{-|x|/a})$$

$$= \rho g \eta_0^2 \frac{a^2}{a^2}\cdot\frac{a}{2} \qquad \eta_x = -\frac{\eta_0}{a} e^{-|x|/a}$$

$$= \frac{1}{2}\rho g a \eta_0^2$$

Initial flow had 0 KE

final flow had $\frac{1}{2}\rho g a\eta_0^2$ KE

- Now look at PE ② (per unit width in y direction) of a block of fluid of height δz length δx at height z (relative to bottom) is

$$[\rho(\delta x . 1 . \delta z)]\, g z$$

Thus is a column with height η has PE $\rho\delta x\, g\int_0^\eta z\,dz$.

So the difference in the final PE from the initial PE, ie. the increase in PE (which we expect to be negative) is $\frac{1}{2}\rho g \eta^2\Big|_{t=\infty}\delta x - \frac{1}{2}\rho g \eta^2\Big|_{t=0}\delta x$

$$= \frac{1}{2}\rho g \delta x\left[\eta^2_{(t=\infty)} - \eta^2_{(t=0)}\right]$$

Thus, over the whole domain, the increase in PE is $2 \cdot \frac{1}{2}\rho g \int_0^\infty \left[\eta_0^2\left(1-e^{-x/a}\right)^2 - \eta_0^2\right]dx$

$$= \rho g \eta_0^2 \int_0^\infty \left(-2e^{-x/a} + e^{-x/2a}\right)dx$$

$$= \rho g \eta_0^2\left[-2 \cdot a + \frac{a}{2}\right]$$

$$= -\frac{3}{2}\rho g a \eta_0^2$$

Thus we have lost, PE equal to $\frac{3}{2}\rho g a \eta_0^2$ (per unit width in y direction) and gained $\frac{1}{2}\rho g a \eta_0^2$ in KE.

→ follows from lin SWE

But energy is conserved (no viscosity). (Can show this from $\frac{\partial}{\partial t}\left(\frac{1}{2}(u^2+v^2)D + \frac{1}{2}g\eta^2\right) = 0$

So an assumption we have made must be incorrect.

To find out, use the full solution. Return to the full problem:

Write $\eta_s(x) = -\eta_0 \operatorname{sgn} x\left[1-e^{-|x|/a}\right]$

Introduce $\bar{\eta}$ as the deviation of η for η_s. (we do this to make the calculation easier)

ie. $\eta = \eta_s + \bar{\eta}$

or $\bar{\eta}(x,t) = \eta(x,t) - \eta_s(x)$

Then $\bar{\eta}(x,0) = \eta(x,0) - \eta_s(x)$
$= -\eta_0 \operatorname{sgn} x + \eta_0 \operatorname{sgn} x\left[1-e^{-|x|/a}\right] = -\eta_0 \operatorname{sgn} x\, e^{-|x|/a}$.

Note both η & η_s satisfy the KG equation & so also does $\bar{\eta}$.

Thus $\left(\frac{\partial^2}{\partial t^2} + f^2 - c^2\frac{\partial^2}{\partial x^2}\right)\bar{\eta} = 0$

Because $\bar{\eta}$ is odd we can use the Fourier Sine transform.

$\bar{\eta}(x,t) = \frac{2}{\pi}\int_0^\infty \bar{\eta}(k,t)\sin kx\, dk$ and as before, find $\bar{\eta}(x,t) = \frac{2}{\pi}\int_0^\infty A(k)\sin kx \cos\omega t\, dk$

Here, the IC is that at $t=0$ $\bar{\eta}(x,0) = -\eta_0 e^{-x/a}$ $x>0$
$= \frac{2}{\pi}\int_0^\infty A(k)\sin kx\, dk$

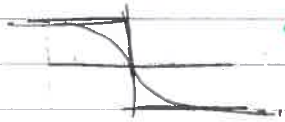
By fourier integral theorem, $A(k) = \int_0^\infty (-\eta_0 e^{-x/a}) \sin kx \, dx$

$$= \frac{-\eta_0 k}{k^2 + \frac{1}{a^2}}$$ FULL SOLUTION.

14/02. Adjustment

Geostrophically balanced - Rossby Gill adjustment problem.

Loss of PE from initial = 3x gain in KE from final.



$$\eta = \eta_s + \bar{\eta}$$

$$\bar{\eta} = \frac{2}{\pi} \int_0^\infty A(k) \sin kx \cos \omega t \, dk.$$

$$A(k) = -\eta_0 k \Big/ \left(k^2 + \frac{1}{a^2}\right)$$

$$\eta(x,t) = \eta_s(x) - \frac{2\eta_0}{\pi} \int_0^\infty \frac{k}{k^2 + \frac{1}{a^2}} \sin kx \cos \omega t \, dk.$$

What is the form of this solution at large t?

$$\int_a^b F(k) e^{i\psi(k)t} \, dk$$ [where in our case $\psi(k) = k\frac{x}{t} - \omega(k)$

& for PW in general, $\psi(k) = k\frac{x}{t} - \sqrt{f^2 + c^2 k^2}$] in lim $t \to \infty$

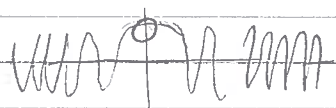
$\kappa = \psi(k)$
$d\kappa = \psi'(k) \, dk$

$$\int_{\kappa=a}^{\kappa=b} F(k(\kappa)) e^{i\kappa t} \frac{d\kappa}{\psi'(k(\kappa))}$$ OK if ψ' not zero in (a,b), ie. single signed ie. $\kappa \leftrightarrow k$ is 1:1.

Then by Riemann-Lebesgue, this behaves as $\frac{1}{t}$ as $t \to \infty$

But what happens if ψ' does vanish in $[a,b]$? see Bender-Orszag handout

Aside: consider $\cos(k t^2)$



So $\int_{-a}^{a} F(t) \cos kt^2 \, dt$ (largest contribution for near 0 as we get cancellation anywhere else)

$$\frac{d}{dt} \cos t^2 = -2t \sin t^2 = 0 \text{ at } t = 0.$$

$x \longleftrightarrow t$ in the following steps as following handout.

$$J(x) \sim f(a)\, e^{\,ix\psi(a) \pm i\pi/2p}\left[\frac{p!}{t|\psi^{(p)}(a)|}\right]^{1/p}\frac{\Gamma(1/p)}{p}$$

won't be asked to derive this formula but might be asked to use it

p = 1st non zero term in expansion.

$\psi'(a) = 0$, $\psi''(a) \neq 0$ $\quad p=2$.

$\psi'(a) = 0$, $\psi''(a) = 0$, $\psi'''(a) \neq 0$ $\quad p=3$.

Now apply this to a wave problem: -

$$\int_{-\infty}^{\infty} A(k)\, e^{i\phi(k)t}\, dk$$ where the phase is $\phi(k) = k\frac{x}{t} - \omega(k)$

so $\phi'(k) = \frac{x}{t} - \omega'(k)$ $\qquad \omega(k)$ dispersion relation.

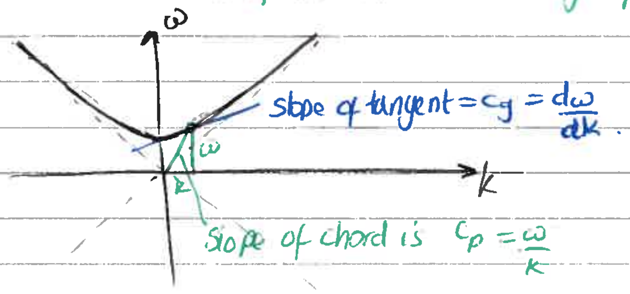
ie. the phase is stationary whenever $\frac{x}{t} = \frac{d\omega}{dk}$

ie stationary on points moving at speed $\frac{d\omega}{dk}$. (Group velocity) (c_g)

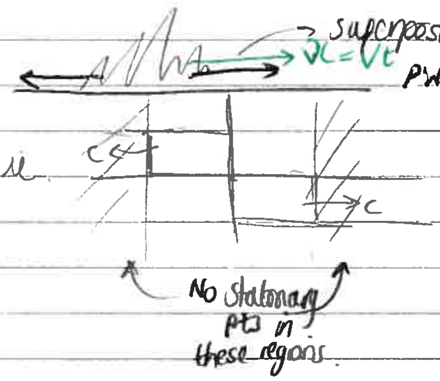
* Thus if we move at the group velocity, c_g, then the solution decays as $\frac{1}{\sqrt{t}}$ but at any other speed it decays as $\frac{1}{t}$. The energy of the motion travels at the group velocity.

[It could actually decay more slowly for some waves eg if c_g vanishes ie $\frac{d^2\omega}{dk^2} = 0$ for some k, since then we have decay like $t^{-1/3}$

- at maximum or minima of the group velocity (AIRYS)



From graph $c_g < c < c_p$



$\forall V > c$

for no k does $c_g(k) = V$

$\frac{x}{t} = c_g(k_s)$

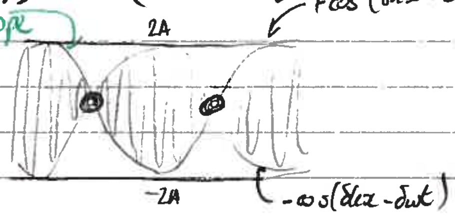
$y_1 = A\cos\left[(k \pm \delta k)x - (\omega \pm \delta\omega)t\right] + \cos(\delta k x - \delta\omega t) = 2A(kx - \omega t)\cos(\delta k x - \delta\omega t)$

envelope travels at $\frac{\delta\omega}{\delta k} = c_g$

crests travel at c_p

$\frac{\delta k}{k} \ll 1$

$\frac{\delta\omega}{\omega} \ll 1$



$\omega = \omega(k)$

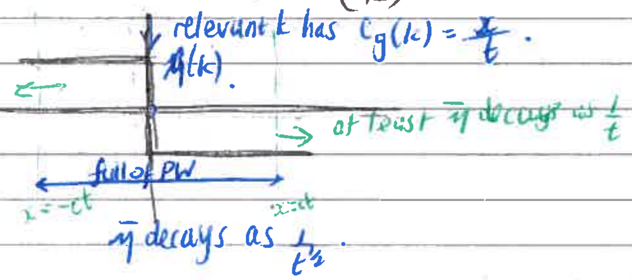
$\frac{\delta\omega}{\delta k} = \omega'(k)$

$\cos\left[k\left(x - \frac{\omega}{k}t\right)\right]\cos\left[\delta k\left(x - \frac{\delta\omega}{\delta k}t\right)\right]$

Ordinary short water waves: $c \propto (g\lambda)^{\frac{1}{2}}$ $\qquad \omega = \beta k^{\frac{1}{2}}$

$\frac{\omega}{k} \propto \left(\frac{g 2\pi}{k}\right)^{\frac{1}{2}}$ $\qquad \frac{\omega}{k} = \frac{\beta}{k^{\frac{1}{2}}}$ $\qquad \frac{d\omega}{dk} = \frac{1}{2}\frac{\beta}{k^{\frac{1}{2}}}$.

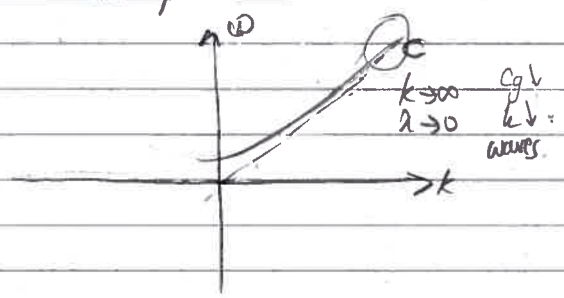
So,

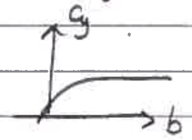


So we have proved that $\bar{\eta}$ decreases to zero everywhere. $\qquad E \propto A^2$

$A \sim \frac{1}{t^{\frac{1}{2}}}$ $\quad E \sim \frac{1}{t}$

$L \to t \Rightarrow EL = \text{constant}$.





$A(\infty) = 0$.

For this problem, $u = \begin{cases} \frac{u_0}{D} c\, J\left(f\sqrt{t^2 - \frac{x^2}{c^2}}\right) & |x| < ct \\ 0 & |x| > ct \end{cases}$

This can also be done in 3 dimensions: (above is in 1D).

The phase $\phi = \underline{k} \cdot \underline{r} - \omega t \quad = kx + ly + mz - \omega t$

$\underline{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$\underline{k} = k\hat{x} + l\hat{y} + m\hat{z}$

The phase is stationary (ie independent of position) when $\nabla\psi = 0$ ie $\frac{\partial k}{\partial x}\hat{x} + \partial l$

$\psi = \frac{\underline{k} \cdot \underline{r}}{t} - \omega$.

$I(x,y,z) = \iiint A(k,l,m)\, e^{i\psi(k,l,m,x,y,z)t}$

ie the gradient in k,l,m space of ψ vanishes ie $\frac{d\psi}{dk} = 0$, $\frac{d\psi}{dl} = 0$, $\frac{d\psi}{dm} = 0$

$\psi = \frac{kx + ly + mz}{t} - \omega$

$\frac{\partial\psi}{\partial k} = 0 \Rightarrow \frac{x}{t} = \frac{\partial\omega}{\partial k}$

$\frac{\partial\psi}{\partial l} = 0 \Rightarrow \frac{y}{t} = \frac{\partial\omega}{\partial l}$

$\frac{\partial\psi}{\partial m} = 0 \Rightarrow \frac{z}{t} = \frac{\partial\omega}{\partial m}$

Thus $\underline{c}_g = \frac{\partial\omega}{\partial k}\hat{x} + \frac{\partial\omega}{\partial l}\hat{y} + \frac{\partial\omega}{\partial m}\hat{z}$.

ie $\underline{c}_g$ is the gradient in wavenumber space of the frequency $\omega(k,l,m)$

[see (Lighthill) waves in fluids]

(For end of chap 2) Handout 14/02

From the rotating Shallow Water Equations to the Conservation of Potential Vorticity

The vector form of the shallow water momentum equation is

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -g\nabla h, \tag{1}$$

where $\boldsymbol{\Omega} = f\hat{\boldsymbol{z}}$ is the constant rotation rate of the frame about the vertical and

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}.\nabla)\boldsymbol{u} \tag{2}$$

$$= \frac{\partial \boldsymbol{u}}{\partial t} + \nabla(\boldsymbol{u}^2/2) + \boldsymbol{\omega} \times \boldsymbol{u}, \tag{3}$$

is the usual advective derivative and $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ is the vorticity. Since $\boldsymbol{u}$ is horizontal and independent of z,

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \zeta\hat{\boldsymbol{z}}. \tag{4}$$

Thus (1) is

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \times \boldsymbol{u} = -\nabla(\boldsymbol{u}^2/2 + gh). \tag{5}$$

Then, since the curl of a gradient vanishes identically, taking the curl of (5) gives

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times [(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \times \boldsymbol{u}] = 0. \tag{6}$$

Now use the identity

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla.\boldsymbol{B}) - \boldsymbol{B}(\nabla.\boldsymbol{A}) + (\boldsymbol{B}.\nabla)\boldsymbol{A} - (\boldsymbol{A}.\nabla)\boldsymbol{B}, \tag{7}$$

with $\boldsymbol{A} = \boldsymbol{\omega} + 2\boldsymbol{\Omega} = (\zeta + f)\hat{\boldsymbol{z}}$ and $\boldsymbol{B} = \boldsymbol{u}$. First note that

$$\nabla.(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) = \nabla.\boldsymbol{\omega} = 0, \tag{8}$$

since $\boldsymbol{\Omega}$ is constant and curl of a divergence is zero. Then note that

$$[(\boldsymbol{\omega} + 2\boldsymbol{\Omega}).\nabla]\boldsymbol{u} = (\zeta + f)(\hat{\boldsymbol{z}}.\nabla)\boldsymbol{u} = (\zeta + f)\frac{\partial \boldsymbol{u}}{\partial z} = 0, \tag{9}$$

since $\boldsymbol{u}$ is independent of z. Thus (6) becomes

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u}.\nabla)(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) + (\boldsymbol{\omega} + 2\boldsymbol{\Omega})\nabla.\boldsymbol{u} = 0. \tag{10}$$

which in fact has only the z component

$$\frac{\partial \zeta}{\partial t} + (\boldsymbol{u}.\nabla)(\zeta + f) + (\zeta + f)\nabla.\boldsymbol{u} = 0, \tag{11}$$

i.e.

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} + (\zeta + f)\nabla.\boldsymbol{u} = 0. \tag{12}$$

Since f is a constant this can be written

$$\frac{\mathrm{D}(\zeta + f)}{\mathrm{D}t} + (\zeta + f)\nabla.\boldsymbol{u} = 0. \tag{13}$$

But the remaining shallow water equation can be written

$$\frac{\mathrm{D}H}{\mathrm{D}t} + H\nabla.\boldsymbol{u} = 0. \tag{14}$$

Taking H times (13) and subtracting $(\zeta + f)$ times (14) gives

$$H\frac{\mathrm{D}(\zeta + f)}{\mathrm{D}t} - (\zeta + f)\frac{\mathrm{D}H}{\mathrm{D}t} = 0, \tag{15}$$

i.e.

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\zeta + f}{H}\right) = 0. \tag{16}$$

Handout 7/02

The Kelvin Wave

The linearised shallow water equations are

$$u_t - fv = -g\eta_x, \tag{1}$$
$$v_t + fu = -g\eta_y, \tag{2}$$
$$\eta_t + D(u_x + v_y) = 0 \tag{3}$$

The roots $\alpha = \pm \mathrm{i}fk/\sigma$ give the solution

$$\eta = \Re\{\eta_0 e^{\pm fky/\sigma} e^{\mathrm{i}(kx-\sigma t)}\} = (Ae^{-fky/\sigma} + Be^{fky/\sigma})\cos(kx - \sigma t) \tag{4}$$

Now $\partial_t(2) - f(1)$ gives

$$v_{tt} + f^2 v = -g\eta_{yt} + fg\eta_x = -2Bfgke^{fky/\sigma}\sin(kx - \sigma t). \tag{5}$$

Notice that the coefficient of A vanishes identically. For v to vanish for all time, and so the left side of (5) to vanish for all t ($\sigma \neq \pm f$), at some fixed point (x, y), e.g. even a single point on the wall $y = 0$, equation (5) implies $B = 0$ and so $v \equiv 0$. A Kelvin wave has zero velocity normal to its supporting wall.

Using $v = 0$ in (1) and (3) and then eliminating u gives

$$\eta_{tt} = c^2 \eta_{xx}, \tag{6}$$

where $c = \sqrt{gD}$, the non-rotating wave equation. For this to have solutions of form (4), $\sigma^2 = c^2k^2$, i.e. $\sigma = \pm ck$. (This is precisely the same result as substituting for α in $\alpha = \pm \mathrm{i}fk/\sigma$.)

If $\sigma = +ck$, then

$$\eta = Ae^{-y/a}\cos[k(x - ct)], \tag{7}$$
$$u = (A/D)ce^{-y/a}\cos[k(x - ct)], \tag{8}$$

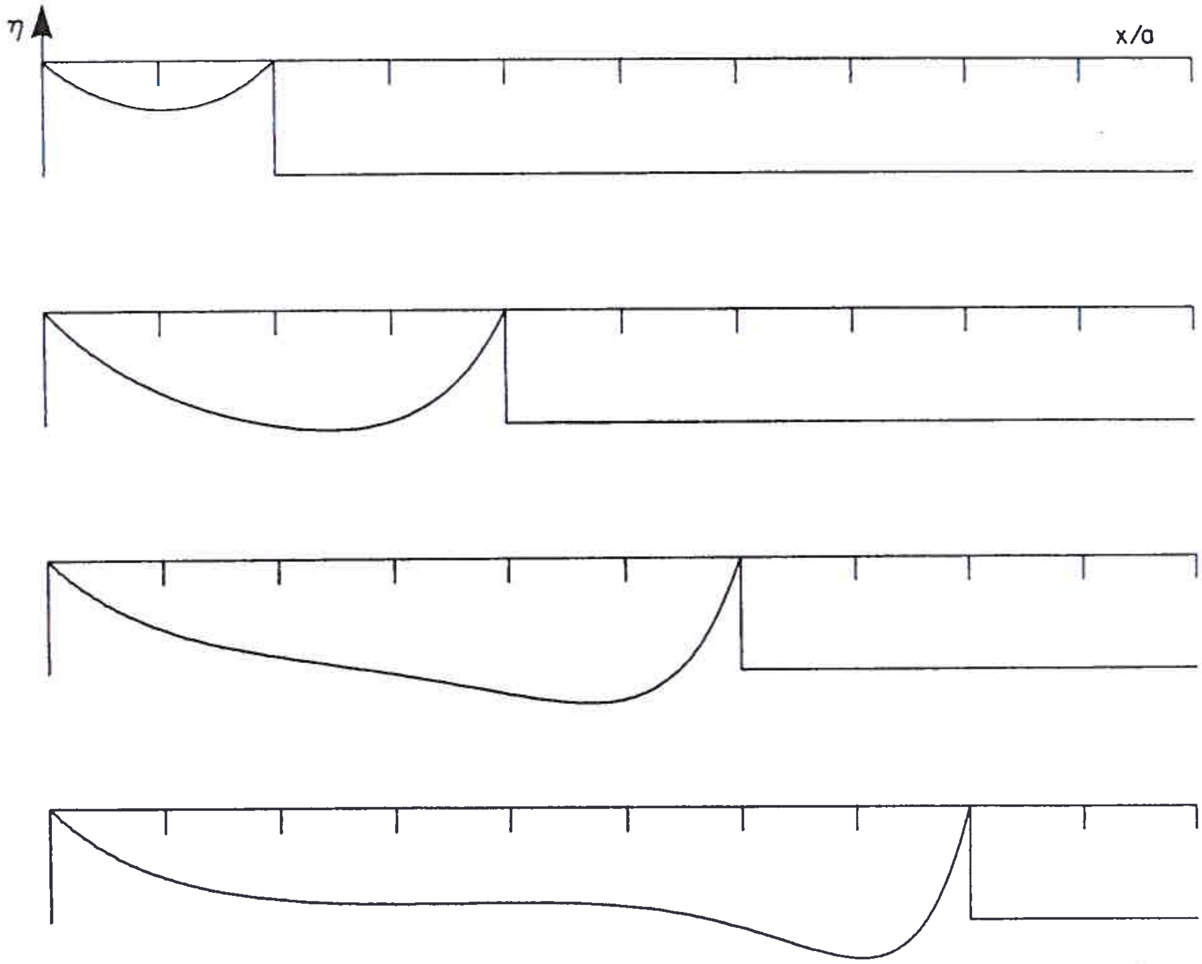
where $a = c/f$ is the Rossby radius and the form of u comes from (2) with $v = 0$:

$$u = (gk/\sigma)Ae^{-fky/\sigma}\cos(kx - \sigma t). \tag{9}$$

If $\sigma = -ck$, then

$$\eta = Ae^{y/a}\cos[k(x + ct)], \tag{10}$$
$$u = -(A/D)ce^{y/a}\cos[k(x + ct)]. \tag{11}$$

For both directions of propagation, the Kelvin wave propagates with it supporting wall to its right and decays exponentially away from the wall on the scale of the Rossby radius.



(a)

Fig. 7.3. Transient profiles for (a) η, (b) u, and (c) v for adjustment under gravity of a fluid with an initial infinitesimal discontinuity in level of $2\eta_0$ at $x = 0$. The solution is shown in the region $x > 0$, where the surface was initially depressed, at time intervals of $2f^{-1}$, where f is twice the rate of rotation of the system about a vertical axis. The marks on the x axis are at intervals of a Rossby radius, i.e., $(gH)^{1/2}/f$, where g is the acceleration due to gravity and H is the depth of fluid. The solutions retain their initial values until the arrival of a wave front that travels out from the position of the initial discontinuity at speed $(gH)^{1/2}$. When the front arrives, the surface elevation rises by η_0 and the u component of velocity rises by $(g/H)^{1/2}\eta_0$ just as in the nonrotating case depicted in Fig. 5.9a. This is because the first waves to arrive are the very short waves, which are unaffected by rotation. Behind the front, however, is a "wake" of waves produced by dispersion, which in the case of u, have the slope given by the Bessel function (7.3.14). This is the point impulse solution to the Klein–Gordon equation. The "width" of the front narrows in inverse proportion with time. Well behind the front, the solution adjusts to the geostrophic equilibrium solution depicted in Fig. 7.1.

Handout 14/02.

provided that $\int_a^b |f(t)|\, dt$ exists. This result is valid even when $f(t)$ is not differentiable and integration by parts or scaling do not work. We will cite the Riemann-Lebesgue lemma repeatedly throughout this section; we could have used it to justify neglecting the integrals on the right sides of (6.5.2) and (6.5.3).

We reserve a proof of the Riemann-Lebesgue lemma for Prob. 6.51. Although the proof of (6.5.6) is messy, it is easy to understand the result heuristically. When x becomes large, the integrand $f(t)e^{ixt}$ oscillates rapidly and contributions from adjacent subintervals nearly cancel.

The Riemann-Lebesgue lemma can be extended to cover generalized Fourier integrals of the form (6.5.1). It states that $I(x) \to 0$ as $x \to +\infty$ so long as $|f(t)|$ is integrable, $\psi(t)$ is continuously differentiable for $a \le t \le b$, and $\psi(t)$ is not constant on any subinterval of $a \le t \le b$ (see Prob. 6.52). The lemma implies that $\int_0^{10} t^3 e^{ix\sin^2 t}\, dt \to 0$ $(x \to +\infty)$, but it does not apply to $\int_0^{10} t^3 e^{2ix}\, dt$.

Integration by parts gives the leading asymptotic behavior as $x \to +\infty$ of generalized Fourier integrals of the form (6.5.1), provided that $f(t)/\psi'(t)$ is smooth for $a \le t \le b$ and nonvanishing at one of the endpoints a or b. Explicitly,

$$I(x) = \frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)} \Big|_{t=a}^{t=b} - \frac{1}{ix}\int_a^b \frac{d}{dt}\frac{f(t)}{\psi'(t)} e^{ix\psi(t)}\, dt.$$

The Riemann-Lebesgue lemma shows that the integral on the right vanishes more rapidly than $1/x$ as $x \to +\infty$. Therefore, $I(x)$ is asymptotic to the boundary terms (assuming that they do not vanish):

$$I(x) \sim \frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)} \Big|_{t=a}^{t=b}, \qquad x \to +\infty. \tag{6.5.7}$$

Observe that when integration by parts applies, $I(x)$ vanishes like $1/x$ as $x \to +\infty$.

Integration by parts may not work if $\psi'(t) = 0$ for some t in the interval $a \le t \le b$. Such a point is called a *stationary* point of ψ. When there are stationary points in the interval $a \le t \le b$, $I(x)$ must still vanish as $x \to +\infty$ by the Riemann-Lebesgue lemma, but $I(x)$ usually vanishes less rapidly than $1/x$ because the integrand $f(t)e^{ix\psi(t)}$ oscillates less rapidly near a stationary point than it does near a point where $\psi'(t) \ne 0$. Consequently, there is less cancellation between adjacent subintervals near the stationary point.

The method of stationary phase gives the *leading* asymptotic behavior of generalized Fourier integrals having stationary points. This method is very similar to Laplace's method in that the leading contribution to $I(x)$ comes from a small interval of width ε surrounding the stationary points of $\psi(t)$. We will show that if c is a stationary point and if $f(c) \ne 0$, then $I(x)$ goes to zero like $x^{-1/2}$ as $x \to +\infty$ if $\psi''(c) \ne 0$, like $x^{-1/3}$ if $\psi''(c) = 0$ but $\psi'''(c) \ne 0$, and so on; as $\psi(t)$ becomes flatter at $t = c$, $I(x)$ vanishes less rapidly as $x \to +\infty$.

Since any generalized Fourier integral can be written as a sum of integrals in which $\psi'(t)$ vanishes only at an endpoint, we can explain the method of stationary phase for the special integral (6.5.1) in which $\psi'(a) = 0$ and $\psi'(t) \ne 0$ for $a < t \le b$.

We decompose $I(x)$ into two terms:

$$I(x) = \int_a^{a+\varepsilon} f(t)e^{ix\psi(t)}\, dt + \int_{a+\varepsilon}^b f(t)e^{ix\psi(t)}\, dt, \tag{6.5.8}$$

where ε is a small positive number to be chosen later. The second integral on the right side of (6.5.8) vanishes like $1/x$ as $x \to +\infty$ because there are no stationary points in the interval $a + \varepsilon \le t \le b$.

To obtain the leading behavior of the first integral on the right side of (6.5.8), we replace $f(t)$ by $f(a)$ and $\psi(t)$ by $\psi(a) + \psi^{(p)}(a)(t-a)^p/p!$ where $\psi^{(p)}(a) \ne 0$ but $\psi'(a) = \cdots = \psi^{(p-1)}(a) = 0$:

$$I(x) \sim \int_a^{a+\varepsilon} f(a) \exp\left\{ix\left[\psi(a) + \frac{1}{p!}\psi^{(p)}(a)(t-a)^p\right]\right\} dt, \qquad x \to +\infty. \tag{6.5.9}$$

Next, we replace ε by ∞, which introduces error terms that vanish like $1/x$ as $x \to +\infty$ and thus may be disregarded, and let $s = (t - a)$:

$$I(x) \sim f(a)e^{ix\psi(a)} \int_0^\infty \exp\left[\frac{ix}{p!}\psi^{(p)}(a)s^p\right] ds, \qquad x \to +\infty. \tag{6.5.10}$$

To evaluate the integral on the right, we rotate the contour of integration from the real-s axis by an angle $\pi/2p$ if $\psi^{(p)}(a) > 0$ and make the substitution

$$s = e^{i\pi/2p}\left[\frac{p!\, u}{x\psi^{(p)}(a)}\right]^{1/p} \tag{6.5.11a}$$

with u real or rotate the contour by an angle $-\pi/2p$ if $\psi^{(p)}(a) < 0$ and make the substitution

$$s = e^{-i\pi/2p}\left[\frac{p!\, u}{x|\psi^{(p)}(a)|}\right]^{1/p}. \tag{6.5.11b}$$

Thus,

$$I(x) \sim f(a)e^{ix\psi(a) \pm i\pi/2p}\left[\frac{p!}{x|\psi^{(p)}(a)|}\right]^{1/p} \frac{\Gamma(1/p)}{p}, \qquad x \to +\infty, \tag{6.5.12}$$

where we use the factor $e^{i\pi/2p}$ if $\psi^{(p)}(a) > 0$ and the factor $e^{-i\pi/2p}$ if $\psi^{(p)}(a) < 0$.

The formula in (6.5.12) gives the leading behavior of $I(x)$ if $f(a) \ne 0$ but $\psi'(a) = 0$. If $f(a)$ vanishes, it is necessary to decide whether the contribution from the stationary point still dominates the leading behavior. When it does, the behavior is slightly more complicated than (6.5.12) (see Prob. 6.53).

Example 3 *Leading behavior of* $\int_0^{\pi/2} e^{ix\cos t}\, dt$ *as* $x \to +\infty$. The function $\psi(t) = \cos t$ has a stationary point at $t = 0$. Since $\psi''(0) = -1$, (6.5.12) with $p = 2$ gives $I(x) \sim \sqrt{\pi/2x}\, e^{i(x-\pi/4)}$ $(x \to +\infty)$.

Bender + Orszag.

Now $f(s) = 1/s$ and $\phi(s) = -s + \ln s$. Laplace's method applies directly to this transformed integral. The maximum of $\phi(s)$ occurs at $s = 1$ so (6.4.19c) gives

$$\Gamma(x) \sim x^x e^{-x}\sqrt{2\pi/x}, \qquad x \to +\infty, \tag{6.4.39}$$

in agreement with (5.4.1). To obtain the next term in the Stirling series we note that $\phi(1) = -1$, $\phi'(1) = 0$, $\phi''(1) = -1$, $\phi'''(1) = 2$, $(d^4\phi/ds^4)(1) = -6$, $f(1) = 1$, $f'(1) = -1$, $f''(1) = 2$. Substituting these coefficients into the formula (6.4.35), we obtain

$$\Gamma(x) \sim x^x e^{-x}\sqrt{\frac{2\pi}{x}}\left(1 + \frac{1}{12x}\right), \qquad x \to +\infty, \tag{6.4.40}$$

in agreement with (5.4.1).

The distinction between ordinary and movable maxima is examined in Probs. 6.45 to 6.47.

(I) 6.5 METHOD OF STATIONARY PHASE

There is an immediate generalization of the Laplace integrals studied in Sec. 6.4 which we obtain by allowing the function $\phi(t)$ in (6.4.1) to be complex. Note that, if we wish, we may assume that $f(t)$ is real; if it were complex, $f(t)$ could be decomposed into a sum of its real and imaginary parts. However, allowing $\phi(t)$ to be complex poses new and nontrivial problems. In this section we consider the special case in which $\phi(t)$ is pure imaginary: $\phi(t) = i\psi(t)$, where $\psi(t)$ is real. The resulting integral

$$I(x) = \int_a^b f(t)e^{ix\psi(t)}\,dt \tag{6.5.1}$$

with $f(t)$, $\psi(t)$, a, b, x all real is called a generalized Fourier integral. When $\psi(t) = t$, $I(x)$ is an ordinary Fourier integral. The general case in which $\phi(t)$ is complex is considered in Sec. 6.6.

To study the behavior of $I(x)$ in (6.5.1) as $x \to +\infty$, we can use integration by parts to develop an asymptotic expansion in inverse powers of x so long as the boundary terms are finite and the resulting integrals exist.

Example 1 *Asymptotic expansion of a Fourier integral as $x \to +\infty$.* We use integration by parts to find an asymptotic approximation to the Fourier integral

$$I(x) = \int_0^1 \frac{e^{ixt}}{1+t}\,dt.$$

After one integration by parts we obtain

$$I(x) = -\frac{i}{2x}e^{ix} + \frac{i}{x} - \frac{i}{x}\int_0^1 \frac{e^{ixt}}{(1+t)^2}\,dt. \tag{6.5.2}$$

The integral on the right side of (6.5.2) is negligible compared with the boundary terms as $x \to +\infty$; in fact, it vanishes like $1/x^2$ as $x \to +\infty$. To see this, we integrate by parts again:

$$-\frac{i}{x}\int_0^1 \frac{e^{ixt}}{(1+t)^2}\,dt = -\frac{1}{4x^2}e^{ix} + \frac{1}{x^2} - \frac{2}{x^2}\int_0^1 \frac{e^{ixt}}{(1+t)^3}\,dt.$$

The integral on the right is bounded because

$$\left|\int_0^1 \frac{e^{ixt}}{(1+t)^3}\,dt\right| \le \int_0^1 (1+t)^{-3}\,dt = \frac{3}{8}.$$

Since the integral on the right in (6.5.2) does vanish like $1/x^2$ as $x \to +\infty$, $I(x)$ is asymptotic to the boundary terms: $I(x) \sim -(i/2x)e^{ix} + i/x$ $(x \to +\infty)$.

Repeated application of integration by parts gives the complete asymptotic expansion of $I(x)$ as $x \to +\infty$: $I(x) = e^{ix}u(x) + v(x)$ where

$$u(x) \sim -\frac{i}{2x} - \frac{1}{4x^2} + \cdots + \frac{(-i)^n(n-1)!}{(2x)^n} + \cdots, \qquad x \to +\infty,$$

$$v(x) \sim \frac{i}{x} + \frac{1}{x^2} + \cdots - \frac{(-i)^n(n-1)!}{x^n} + \cdots, \qquad x \to +\infty.$$

Example 2 *Integration by parts applied to $\int_0^1 \sqrt{t}\,e^{ixt}\,dt$.* Integration by parts can be used just once for the Fourier integral $I(x) = \int_0^1 \sqrt{t}\,e^{ixt}\,dt$. One integration by parts gives

$$I(x) = -\frac{i}{x}e^{ix} + \frac{i}{2x}\int_0^1 \frac{e^{ixt}}{\sqrt{t}}\,dt. \tag{6.5.3}$$

The integral on the right side of (6.5.3) vanishes more rapidly than the boundary term as $x \to +\infty$. We cannot use integration by parts to verify this because the resulting integral does not exist. (Why?) However, we can use the following simple scaling argument. We let $s = xt$ and obtain

$$\frac{i}{2x}\int_0^1 \frac{e^{ixt}}{\sqrt{t}}\,dt = \frac{i}{2x^{3/2}}\int_0^x \frac{e^{is}}{\sqrt{s}}\,ds \sim \frac{i}{2x^{3/2}}\int_0^\infty \frac{e^{is}}{\sqrt{s}}\,ds, \qquad x \to +\infty.$$

To evaluate the last integral we rotate the contour of integration from the real-s axis to the positive imaginary-s axis in the complex-s plane and obtain

$$\int_0^\infty \frac{e^{is}}{\sqrt{s}}\,ds = \sqrt{\pi}\,e^{i\pi/4}. \tag{6.5.4}$$

(See Prob. 6.49 for the details of this calculation.) Therefore,

$$I(x) + \frac{i}{x}e^{ix} \sim \frac{i}{2x^{3/2}}\sqrt{\pi}\,e^{i\pi/4}, \qquad x \to +\infty. \tag{6.5.5}$$

Clearly, this result cannot be found by direct integration by parts of the integral on the right side of (6.5.3) because a fractional power of x has appeared. However, it is possible to find the full asymptotic expansion of $I(x)$ as $x \to +\infty$ by an indirect application of integration by parts (see Prob. 6.50).

In Example 1 we used integration by parts to argue that the integral on the right side of (6.5.2) vanishes more rapidly than the boundary terms as $x \to +\infty$. In Example 2 we used a scaling argument to show that the integral on the right side of (6.5.3) vanishes more rapidly than the boundary terms as $x \to +\infty$. There is, in fact, a very general result called the Riemann-Lebesgue lemma that guarantees that

$$\int_a^b f(t)e^{ixt}\,dt \to 0, \qquad x \to +\infty, \tag{6.5.6}$$

Example 4 *Leading behavior of* $\int_0^\infty \cos(xt^2 - t)\,dt$ *as* $x \to +\infty$. To use the method of stationary phase, we write this integral as $\int_0^\infty \cos(xt^2 - t)\,dt = \mathrm{Re}\int_0^\infty e^{i(xt^2-t)}\,dt$. The function $\psi(t) = t^2$ has a stationary point at $t = 0$. Since $\psi''(0) = 2$, (6.5.12) with $p = 2$ gives $\int_0^\infty \cos(xt^2 - t)\,dt \sim \mathrm{Re}\,\frac{1}{2}\sqrt{\pi/x}\,e^{i\pi/4} = \frac{1}{2}\sqrt{\pi/2x}$ $(x \to +\infty)$.

Example 5 *Leading behavior of* $J_n(n)$ *as* $n \to \infty$. When n is an integer, the Bessel function $J_n(x)$ has the integral representation

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x \sin t - nt)\,dt \tag{6.5.13}$$

(see Prob. 6.54). Therefore, $J_n(n) = \mathrm{Re}\int_0^\pi e^{in(\sin t - t)}\,dt/\pi$. The function $\psi(t) = \sin t - t$ has a stationary point at $t = 0$. Since $\psi''(0) = 0$, $\psi'''(0) = -1$, (6.5.12) with $p = 3$ gives

$$\begin{aligned} J_n(n) &\sim \frac{1}{\pi}\mathrm{Re}\left[\frac{1}{3}e^{-i\pi/6}\left(\frac{6}{n}\right)^{1/3}\Gamma\left(\frac{1}{3}\right)\right], \qquad x \to +\infty, \\ &= \frac{1}{\pi}2^{-2/3}3^{-1/6}\Gamma\left(\frac{1}{3}\right)n^{-1/3}, \qquad n \to \infty. \end{aligned} \tag{6.5.14}$$

Observe that because $\psi''(0) = 0$, $J_n(n)$ vanishes less rapidly than $n^{-1/2}$ as $n \to \infty$.

If n is not an integer, (6.5.14) still holds (see Prob. 6.55).

In this section we have obtained only the leading behavior of generalized Fourier integrals. Higher-order approximations can be complicated because nonstationary points may also contribute to the large-x behavior of the integral. Specifically, the second integral on the right in (6.5.8) must be taken into account when computing higher-order terms because the error incurred in neglecting this integral is usually algebraically small. By contrast, recall that the approximation in (6.4.2) for Laplace's method is valid to all orders because the errors are exponentially, rather than algebraically, small. To obtain the higher-order corrections to (6.5.12), one can either use the method of asymptotic matching (see Sec. 7.4) or the method of steepest descents (see Sec. 6.6).

(I) 6.6 METHOD OF STEEPEST DESCENTS

The method of steepest descents is a technique for finding the asymptotic behavior of integrals of the form

$$I(x) = \int_C h(t)e^{x\rho(t)}\,dt \tag{6.6.1}$$

as $x \to +\infty$, where C is an integration contour in the complex-t plane and $h(t)$ and $\rho(t)$ are analytic functions of t. The idea of the method is to use the analyticity of the integrand to justify deforming the contour C to a new contour C' on which $\rho(t)$ has a constant imaginary part. Once this has been done, $I(x)$ may be evaluated asymptotically as $x \to +\infty$ using Laplace's method. To see why, observe that on the contour C' we may write $\rho(t) = \phi(t) + i\psi$, where ψ is a real constant and $\phi(t)$ is a real function. Thus, $I(x)$ in (6.6.1) takes the form

$$I(x) = e^{ix\psi}\int_{C'} h(t)e^{x\phi(t)}\,dt. \tag{6.6.2}$$

Although t is complex, (6.6.2) can be treated by Laplace's method as $x \to +\infty$ because $\phi(t)$ is real.

Our motivation for deforming C into a path C' on which $\mathrm{Im}\,\rho(t)$ is a constant is to eliminate rapid oscillations of the integrand when x is large. Of course, one could also deform C into a path on which $\mathrm{Re}\,\rho(t)$ is a constant and then apply the method of stationary phase. However, we have seen that Laplace's method is a much better approximation scheme than the method of stationary phase because the full asymptotic expansion of a generalized Laplace integral is determined by the integrand in an arbitrarily small neighborhood of the point where $\mathrm{Re}\,\rho(t)$ is a maximum on the contour. By contrast, the full asymptotic expansion of a generalized Fourier integral typically depends on the behavior of the integrand along the entire contour. As a consequence, it is usually easier to obtain the full asymptotic expansion of a generalized Laplace integral than of a generalized Fourier integral.

Before giving a formal exposition of the method of steepest descents, we consider three preliminary examples which illustrate how shifting complex contours can greatly simplify asymptotic analysis. In the first example we consider a Fourier integral whose asymptotic expansion is difficult to find by the methods used in Sec. 6.5. However, deforming the contour reduces the integral to a pair of integrals that are easy to evaluate by Laplace's method.

Example 1 *Conversion of a Fourier integral into a Laplace integral by deforming the contour.* The behavior of the integral

$$I(x) = \int_0^1 \ln t\, e^{ixt}\,dt \tag{6.6.3}$$

as $x \to +\infty$ cannot be found directly by the methods of Sec. 6.5 because there is no stationary point. Also, integration by parts is useless because $\ln 0 = -\infty$. Integration by parts is doomed to fail because, as we will see, the leading asymptotic behavior of $I(x)$ contains the factor $\ln x$ which is not a power of $1/x$.

To approximate $I(x)$ we deform the integration contour C, which runs from 0 to 1 along the real-t axis, to one which consists of three line segments: C_1, which runs up the imaginary-t axis from 0 to iT; C_2, which runs parallel to the real-t axis from iT to $1 + iT$; and C_3, which runs down from $1 + iT$ to 1 along a straight line parallel to the imaginary-t axis (see Fig. 6.5). By Cauchy's theorem, $I(x) = \int_{C_1+C_2+C_3} \ln t\, e^{ixt}\,dt$. Next we let $T \to +\infty$. In this limit the contribution from C_2 approaches 0. (Why?) In the integral along C_1 we set $t = is$, and in the integral along C_3 we set $t = 1 + is$, where s is real in both integrals. This gives

$$I(x) = i\int_0^\infty \ln(is)\,e^{-xs}\,ds - i\int_0^\infty \ln(1 + is)\,e^{ix(1+is)}\,ds. \tag{6.6.4}$$

The sign of the second integral on the right is negative because C_3 is traversed downward.

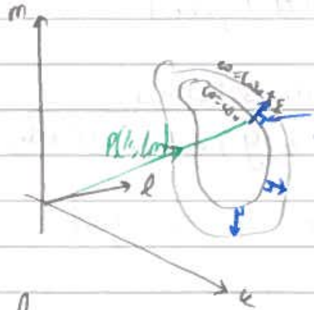
Observe that both integrals in (6.6.4) are Laplace integrals. The first integral can be done exactly. We substitute $u = xs$ and use $\ln(is) = \ln s + i\pi/2$ and the identity $\int_0^\infty e^{-u}\ln u\,du = -\gamma$, where $\gamma = 0.5772\ldots$ is Euler's constant, and obtain

$$i\int_0^\infty \ln(is)\,e^{-xs}\,ds = -i(\ln x)/x - (i\gamma + \pi/2)/x.$$

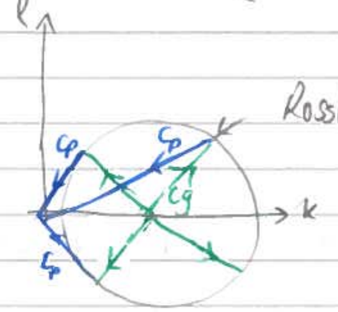
We apply Watson's lemma to the second integral on the right in (6.6.4) using the Taylor expansion $\ln(1 + is) = -\sum_{n=1}^\infty (-is)^n/n$, and obtain

$$-i\int_0^\infty \ln(1 + is)\,e^{ix(1+is)}\,ds \sim ie^{ix}\sum_{n=1}^\infty \frac{(-i)^n(n-1)!}{x^{n+1}}, \qquad x \to +\infty.$$

We call a surface on which the frequency is constant a SLOWNESS surface.
i.e. surface $\omega(k,l,m)=\omega_0$ for some ω_0 (fixed).



(c_g) group velocity (must be orthogonal to slowness surfaces & in the direction of ω increasing)



Rossby waves slowness surface is a circle

PW: $\omega^2 = f^2 + c^2k^2$ $\quad\quad \frac{\omega^2}{k^2} = c^2 + \frac{f^2}{k^2} > c^2$

$$2\omega \frac{d\omega}{dk} = 2c^2 k$$

$$\frac{\omega}{k}\frac{d\omega}{dk} = c^2$$

$$c_p c_g = c^2$$

End of term 1 $\quad |c_p| > c$ so $|c_g| < c \Rightarrow c_g < c_p$.

Chapter 3?

25/02 lrSWE
$$u_t - fv = -g\eta_x \quad ①$$
$$v_t + fu = -g\eta_y \quad ②$$
$$\eta_t + \nabla\cdot(H_0(x,y)\underline{u}) = 0. \quad ③$$

Cross differentiate ① & ②: $\boxed{\left(\frac{\partial^2}{\partial t^2} + f^2\right)\underline{u} = -g\nabla\eta_t + gf\,\hat{\underline{z}}\wedge\underline{\nabla}\eta}$ (as when H_0 = const)

$$(\partial_{tt}+f^2)(3) \Rightarrow (\partial_{tt}+f^2)\eta_t + \underline{\nabla}\cdot\left[H_0(x,y)\underbrace{\{-g\,\underline{\nabla}\eta_t + gf\,\hat{\underline{z}}\wedge\underline{\nabla}\eta\}}_{\underline{V}}\right] = 0$$

$$\underline{\nabla}\cdot(\phi\underline{u}) \equiv \phi\,\underline{\nabla}\cdot\underline{u} + (\underline{u}\cdot\underline{\nabla})\phi \quad \text{for } \forall\underline{u},\phi$$

$$(\partial_{tt}+f^2)\eta_t + H_0(x,y)(-g\nabla^2\eta_t) + (-g\underline{\nabla}\eta_t\cdot\underline{\nabla}H_0) + gf(\hat{\underline{z}}\wedge\underline{\nabla}\eta)\cdot\underline{\nabla}H_0 = 0$$

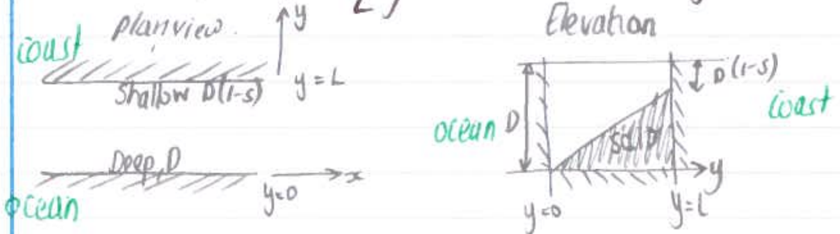
$$(\hat{\underline{z}}\wedge\underline{\nabla}\eta)\cdot\underline{\nabla}H_0 = \hat{\underline{z}}\cdot(\underline{\nabla}\eta\wedge\underline{\nabla}H_0)$$
$$= \frac{\partial\eta}{\partial x}\frac{\partial H_0}{\partial y} - \frac{\partial\eta}{\partial y}\frac{\partial H_0}{\partial x} = \frac{\partial(\eta,H_0)}{\partial(x,y)} \quad \text{(Jacobian)}.$$

Thus we have
$$\underbrace{(\partial_{tt}+f^2)\eta_t - g\underline{\nabla}\cdot(H_0(x,y)\underline{\nabla}\eta_t)}_{\text{Standard wave equation with non constant speed plus rotation}} + \underbrace{gf\frac{\partial(\eta,H_0)}{\partial(x,y)}}_{\text{slope term}} = 0$$

When H_0 = constant, last term is absent and can integrate wrt t to get 2nd order (in t) equation: wave equation.
Now we cannot do that.

Hence the equation is 3rd order in t, ie 3 waves (cubic for ω). Hence we have a new wave.
It requires slope and rotation (we have $\therefore$ not seen it before)

There is no general solution. We will seek a particular solution.
We will consider a linear slope (as a local model of general, slowly-varying topography)
Thus take $H_0 = D\left(1 - \frac{sy}{L}\right)$ in the channel of width L ie $0 \le y \le L$.



Notice this introduces a wavespeed $c^2 = gD$. and our equation becomes

To solve $\left[\right.$ $(\partial_{tt} + f^2)\eta_t - c^2 \nabla \cdot \left[\left(1 - s\frac{y}{L}\right)\nabla \eta_t\right] + c^2 f \frac{\partial \eta}{\partial x}\left(-\frac{s}{L}\right) = 0$ $\quad \frac{\partial H_0}{\partial x} = 0$

The BCs on $y = 0, L$ are $v = 0$ as before
i.e. $\eta_{yt} - f\eta_x = 0$ on $y = 0, L$

[This can be solved elegantly by Laplace Transforms in y]

But in fact, the interesting dynamics are driven by the final term, thus it is sufficient to take $0 < s \ll 1$. Then $1 - s\frac{y}{L} \sim 1$, to leading order in S.

We can still look for a solution of the form $\eta(x,y,t) = \mathrm{Re}\{\bar{\eta}(y) e^{i(kx - \omega t)}\}$.
Substitution gives:
$$(f^2 - \omega^2)(-i\omega)\bar{\eta} - i\omega k^2 c^2\left(1 - s\frac{y}{L}\right)\bar{\eta} - i\omega c^2 \frac{d}{dy}\left[\left(1 - s\frac{y}{L}\right) \times \frac{d\bar{\eta}}{dy}\right] + ikf\bar{\eta}c^2\left(-\frac{s}{L}\right) = 0$$
Rearranging and taking $s \ll 1$:
$$\bar{\eta}'' \cancel{- \frac{s}{L}\bar{\eta}'} + \bar{\eta}\left[\frac{\omega^2 - f^2}{c^2} - k^2 - \frac{fsk}{L\omega}\right] = 0.$$
0 (show in notes why).

Then this is exactly our previous problem.
$$\bar{\eta}' + \alpha^2 \bar{\eta} = 0 \quad \text{with however} \quad \alpha^2 = \underbrace{\frac{\omega^2 - f^2}{c^2} - k^2}_{\text{as before}} - \underbrace{\frac{fsk}{L\omega}}_{\text{new term}}.$$
NB. if the slope or rotation disappears, the new term is absent ie. if $f = 0$, $s = 0$

Cubic (has 3 terms)

28/02 Same equation. $\eta(x,y,t) = \mathrm{Re}\{\bar\eta(y) e^{i(kx-\omega t)}\}$. where $\bar\eta$ satisfies

$$\bar\eta'' + \alpha^2 \bar\eta = 0 \qquad \alpha^2 = \frac{\omega^2 - f^2}{c^2} - k^2 - \frac{fsk}{L\omega}$$

$\omega\bar\eta' + kf\bar\eta = 0 \quad y = 0, L$ (as before).

As before, can write $\bar\eta = A\cos\alpha y + B\sin\alpha y$.

Apply the BC's $\underbrace{(\omega^2 - f^2)(\omega^2 - c^2k^2)}_{2\ k\ \text{waves}}\underbrace{\sin\alpha L}_{PW's} = 0$ as before.

The change to the kelvin waves for $0<s\ll1$ is order s, ie small.

Consider $\sin\alpha L = 0$ ie. $\alpha L = n\pi$, $n = 1,2,3,\dots$

$\alpha^2 = \frac{n^2\pi^2}{L^2}$. Then $\omega^2 = \underbrace{c^2(k^2+\alpha^2) + f^2}_{\text{usual PW's. } (s=0)} + \frac{fsk}{L\omega}c^2$

Formally, this is a cubic in ω ie. 3 waves for each k (2 PW's and a new wave).
But $0<s\ll1$. Put $s=0$.
$\omega_0^2 = c^2(k^2+\alpha^2) + f^2$ ie. usual PW's.

Now suppose $0<s\ll1$ $\quad \omega^2 = c^2(k^2+\alpha^2) + f^2 + \frac{fsk}{L}c^2 \cdot \frac{1}{\omega_0}$

(accurate to order s)

by replacing ω by ω_0 make error order s
But you multiply by s so error is of order s^2

This is boring as we see a small change in PWs due to the small s.

Where is the new wave?
The third root is of order s. ie. for third root, $\omega\to0$ as $s\to0$ with $\frac{\omega}{s}$ fixed

Then $0 = c^2(k^2+\alpha^2) + f^2 + \frac{fsk}{L\omega}c^2$

ie $\frac{fsk}{L\omega} = -\left[k^2+\alpha^2+\frac{f^2}{c^2}\right]$

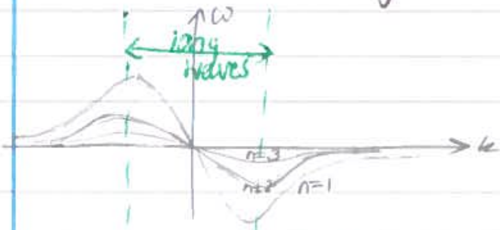
ie $\omega = \frac{-fsk/L}{k^2 + \frac{n^2\pi^2}{L^2} + \frac{1}{a^2}}$ $\qquad a = \frac{c}{f}$ Rossby Radius.

$$\omega = \frac{-fsk/L}{\underset{[L^{-2}]}{k^2} + \underset{[L^{-2}]}{\frac{n^2\pi^2}{L^2}} + \underset{[L^{-2}]}{\frac{1}{a^2}}}$$

Need $s\neq0$ and $f\neq0$.

— a (topographic) Rossby wave.

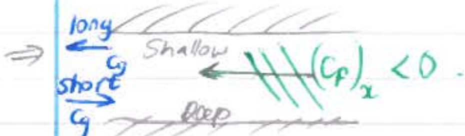
Note for $0 < s \ll 1$, $\frac{\omega}{f} \ll 1$. Low frequency waves. → periods of many days.



max at $k = \sqrt{\frac{n^2\pi^2}{L^2} + \frac{1}{a^2}}$

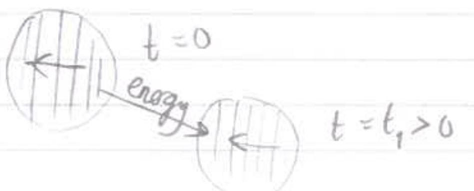
NB: $\frac{\omega}{k} = \frac{-fs/L}{k^2 + \frac{n^2\pi^2}{L^2} + \frac{1}{a^2}}$ $(= c_p) < 0 \ \forall k$ ie all crests travel to the left ie. with shallow water to their right. (as for KWs)

KWs always have the wall on their right.



$c_g < 0$ for long waves ie $|k| < \sqrt{\frac{n^2\pi^2}{L^2} + \frac{1}{a^2}}$ ie between the maxima.

$c_g > 0$ for short waves.



$t = t_1 > 0$ time increasing

* Rotating flow film on fluids portal *

So far:

SWE

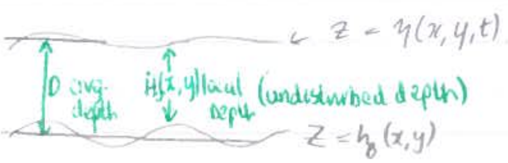
3 waves: 2 (PW + KW) → high freq $\omega \sim f$ (hrs)

1 (RW) → low freq $\omega \ll f$ (days).

Need to look for a limit that filters out PWs (+ high freq KWs), but keeps the RWs

ie. we take $\frac{\omega}{f} \ll 1$ (but do not linearise).

(2.11) The Quasi-geostrophic limit (of SWE).



so $H_0(x,y) + h_B = D$.

Non dimensionalise quantities. Write $(x,y) = L\,(x', y')$

L: typical horizontal length; (x', y'): numbers. - no dimensions.

$$t = T t'$$
$$(u,v) = U(u', v')$$
$$\eta = N_0 \eta'.$$

$$H\left(\frac{x}{L},\frac{y}{L},\frac{t}{T}\right) = H_0(x',y') + \eta(x',y',t')$$
$$= D - h_B(x',y') + \eta(x',y',t')$$
$$= D\left[1 - \frac{h_B}{D} + \frac{N_0}{D}\eta'\right] \longleftarrow$$ all terms here are dimensionless $\left(\frac{\text{length}}{\text{length}}\right)$

when we divide by Uf →

$$\underbrace{\overset{\varepsilon_T}{\frac{U}{T}}\frac{\partial u'}{\partial t'}}_{\text{time}} + \overset{\varepsilon}{\frac{U^2}{L}}\underbrace{\left(u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'}\right)}_{\text{non linear}} - \underbrace{\overset{1}{Uf}\,v'}_{\text{coriolis}} = \underbrace{-\overset{1}{\frac{gN_0}{L}}\frac{\partial \eta'}{\partial x'}}_{\text{pressure}}$$

$$\overset{\varepsilon_T}{\frac{U}{T}}\frac{\partial v'}{\partial t'} + \overset{\varepsilon}{\frac{U^2}{L}}\left(u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'}\right) + \underset{1}{Uf}\,u' = -\underset{1}{\frac{gN_0}{L}}\frac{\partial \eta'}{\partial y'}$$

when we divide by $\frac{U}{LD}$ →

$$\underset{\varepsilon_T F}{\frac{N_0}{T}}\frac{\partial \eta'}{\partial t'} + \underset{\varepsilon F}{\frac{U}{L}}\left[u'\frac{\partial}{\partial x'}\left(\cancel{N_0}\eta' - \frac{h_B}{D}\right) + v'\frac{\partial}{\partial y'}\left(\cancel{N_0}\eta' - \frac{h_B}{D}\right)\right]$$
$$+ \frac{U}{L}\left(\underset{1}{D} + \underset{\varepsilon F}{N_0\eta'} - \frac{h_B}{D}\right)\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) = 0$$

$$\left[\frac{\partial \eta}{\partial t} + \underline{u}\cdot\nabla H + H\nabla\cdot\underline{u} = 0\right]$$

Choose N_0 so that the leading order balance is geostrophic, ie, $g\frac{N_0}{L} = Uf$.

ie. $N_0 = \frac{UfL}{g}$.

∴ The ratio of non-linear to Coriolis is $\frac{U^2/L}{U/F} = \frac{U}{fL} = \varepsilon$, Rossby number (nondimensional)

Eventually take ε to be small, consistent with leading order behaviour geostrophic

The ratio of time changes in u' to coriolis is $\frac{U/T}{Uf} = \frac{1}{fT} = \varepsilon_T$,

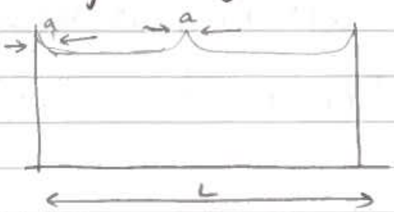
a temporal Rossby number (eventually $0 < \varepsilon_T \ll 1$).

With these note that $\frac{N_0}{D} = \frac{UfL}{D} = \frac{UfL}{c^2} = \frac{U}{fL}\left(\frac{f^2}{c^2}\right)L^2 = \varepsilon\left(\frac{L}{a}\right)^2 \longleftarrow$ Rossby radius.

Traditionally written (Pedlosky) $F = \frac{L^2}{a^2}$.

– ratio of the basin of the Rossby radius

If $F \gg 1$, basin is many Rossby radii across.



Disturbances are confined to distances of order a

$\frac{a}{L} \ll 1$.

Or $F \ll 1$, $\frac{a}{L} \gg 1$ $(g \to \infty)$ Surface's effectively rigid

– disturbances not confined.

Rigid-lid limit.

We keep $F \sim 1$ so we can treat all F.

Take ε, ε_T as our small parameters.

Three cases : 1. $1 \gg \varepsilon_T \gg \varepsilon$: non linearity $\ll$ time dependance
- the linear SWE
- already solved

2. $1 \gg \varepsilon_T \sim \varepsilon$: non linearity but no PWS $\left(\frac{\omega}{f} \sim \varepsilon_T \ll 1\right)$

3. $1 \gg \varepsilon \gg \varepsilon_T$: steady non linear (contained in 2).

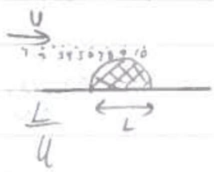
Thus sufficient to consider 2. ie. we take $\boxed{\varepsilon_T = \varepsilon}$.

ie. $\frac{1}{fT} = \frac{U}{fL}$

ie. $T = \frac{L}{U}$. ie. the advection time

This is much longer than the rotational period $\frac{1}{f}$

$$L/U \Big/ \tfrac{1}{f} = fL/U = \varepsilon^{-1} \gg 1.$$



ie. Earth spins many times when a particle crosses a bump eg (10 times).

$\therefore$ Dropping the dashes (but we are still in non dimensional coordinates), we have

$$(1)\quad \varepsilon\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) - v = -\frac{\partial \eta}{\partial x}$$

$$(2)\quad \varepsilon\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + u = -\frac{\partial \eta}{\partial y}$$

$$(3)\quad \varepsilon F\left(\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}\right) - \left(u\frac{\partial h_B/D}{\partial x} - v\frac{\partial h_B/D}{\partial y}\right) + \left(1 + \varepsilon F\eta - \frac{h_B}{D}\right)\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{O(\varepsilon)} = 0$$

Any solution of this would be of the form $u(x,y,t;\varepsilon,F)$.

Suppose ε is small and this sol^n has power series expansion in ε.

ie. $u(x,y,t;\varepsilon,F) = u^{(0)}(x,y,t;F) + \varepsilon u^{(1)}(x,y,t;F) + \varepsilon^2 u^{(2)}(x,y,t;F) + \dots$

Similarly for v, η.

Now substitute into (1), (2), (3) and equate powers of ε.

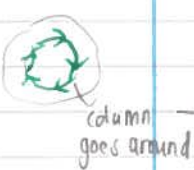
Lowest order of ε in (1) is ε^0: $[\varepsilon^0]: -v^{(0)} = -\frac{\partial \eta^{(0)}}{\partial x}$

In (2) $[\varepsilon^0]: u^{(0)} = -\frac{\partial \eta^{(0)}}{\partial y}$.

no surprise - we insisted that leading order balance is geostrophic.

Equation (3): leading order term depends on size of $\frac{h_B}{D}$

i.e. the fractional change in depth due to bottom topography.



Suppose $\frac{h_B}{D} \sim \varepsilon^0$ i.e. $O(1)$ - large depth changes.

$$[\varepsilon^0] \quad u^{(0)} \frac{\partial}{\partial x}\left(\frac{h_B}{D}\right) + v^{(0)} \frac{\partial}{\partial y}\left(\frac{h_B}{D}\right) = 0$$

NB : $\frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} = 0$.

So leading order terms of $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ is $\varepsilon\left(\frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y}\right)$

$$\underline{u}^{(0)} \cdot \nabla\left(\frac{h_B}{D}\right) = 0$$

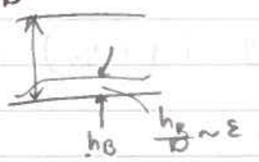
$$\underline{u}^{(0)} \perp^r \nabla\left(\frac{h_B}{D}\right.$$

i.e. $\underline{u}^{(0)}$ // to level curves of $\frac{h_B}{D}$. i.e. flow is confined to bottom contours - only flow.

Flow follows bathymetry because it is so strong

4/03. $\quad u^{(0)} = \frac{\partial \eta^{(0)}}{\partial y} \qquad v^{(0)} = \frac{\partial \eta^{(0)}}{\partial x} \qquad \left(\text{So } \frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} = 0\right)$

Last time: large topography. $\frac{h_B}{D} \sim 1$: flow follows isobaths.
This time: bring all of (3) into play by considering small topography.
$\frac{h_B}{D} \sim \varepsilon$. we write $\frac{h_B(x,y)}{D} = \varepsilon \eta_B(x,y)$ where η_B is order unity.



Where we have $\varepsilon(u_t + uu_x + v_y) - v = -\eta_x$ (1)

$$\varepsilon(v_t + uv_x + vv_y) + u = -\eta_y \qquad (2)$$

$$\varepsilon F(\eta_t + u\eta_x + v\eta_y) - \varepsilon\left(u\frac{\partial}{\partial x}\eta_B + v\frac{\partial \eta_B}{\partial y}\right.$$

$$+ (1 + \varepsilon F\eta - \varepsilon\eta_B)\underbrace{(u_x + v_y)}_{o(\varepsilon)\text{ term.}} = 0 \qquad (3)$$

$$\varepsilon\left(\frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y}\right).$$

Now the leading order terms in (3) are

$$[\varepsilon] : F\frac{D_0\eta^{(0)}}{Dt} - \frac{D_0\eta_B}{Dt} + \frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} = 0 \qquad \frac{D_0}{Dt} = \frac{\partial}{\partial t} + u^{(0)}\frac{\partial}{\partial x} + v^{(0)}\frac{\partial}{\partial y}$$

$$\frac{D_0}{Dt}\left[F\eta^{(0)} - \eta_B\right] + \frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} = 0. \quad (4)$$

$$(1) : [\varepsilon] \quad \frac{D_0 u^{(0)}}{Dt} - v^{(1)} = -\frac{\partial \eta^{(1)}}{\partial x}$$

$$(2) : \quad \frac{D_0 v^{(0)}}{Dt} + u^{(1)} = -\frac{\partial \eta^{(1)}}{\partial y}$$

$$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1) : \frac{D_0}{Dt}\zeta^{(0)} + \frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} = 0 \quad (5) \qquad \zeta^{(0)} = \frac{\partial v^{(0)}}{\partial x} - \frac{\partial u^{(0)}}{\partial y}.$$

(5) - (4) gives $\frac{D_0}{Dt}\left(\zeta^{(0)} - F\eta^{(0)} + \eta_B\right) = 0.$

This is a closed zero-order system.

$$u^{(0)} = -\frac{\partial \eta^{(0)}}{\partial y} \qquad v^{(0)} = \frac{\partial \eta^{(0)}}{\partial x} \qquad \zeta^{(0)} = \frac{\partial^2 \eta^{(0)}}{\partial x^2} + \frac{\partial^2 \eta^{(0)}}{\partial y^2} = \nabla^2 \eta^{(0)}$$

Notice $\eta^{(0)}$ behaves like a streamfunction.
Write $\psi = \eta^{(0)}$ ← surface displacement

$$\frac{D_0}{Dt} = \frac{\partial}{\partial t} + \underbrace{\frac{\partial(\psi, \cdot)}{\partial(x,y)}}_{\text{Jacobian}}$$

→ actually geostrophic, though.

Thus we get the quasi geostrophic equations $u = -\psi_y \quad v = \psi_x \quad \zeta = \nabla^2\psi$.
GFD DEFINITION

(Careful, this is identical to inviscid, incompressible 2D dynamics (Math 2301) <u>except</u> ψ has been multiplied by -1).

non linear.

The governing equation is then, $\left[\frac{\partial}{\partial t} + \frac{\partial(\psi, \cdot)}{\partial(x,y)}\right]\left(\nabla^2\psi - F\psi + \eta_B\right) = 0$
(QG).

ie. $\frac{Dq}{Dt} = 0$. where $q = \nabla^2\psi - F\psi + \eta_B$. – The Quasi Geostrophic equation for the conservation of PV.
QGPV.

- a very powerful and useful set of equations.
- low frequency limit of the rotating shallow water equations.
- not linear
- no poincaré waves
- they will have the Rossby waves & the remnant of the KWs.

Return to the channel model with the sloping bottom.

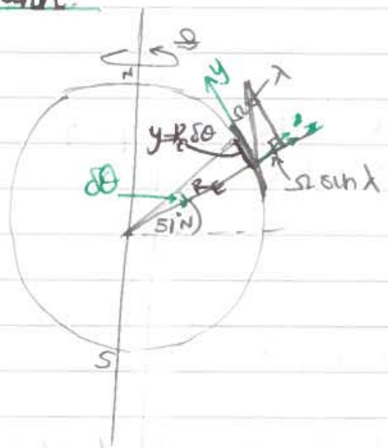
$H_b(x,y) = D\left(1 - s\frac{y^*}{L}\right)$ $\quad$ * ⇒ dimensional.

This is precisely $\eta_B = \beta y$; where $\beta = \frac{s}{\varepsilon}$. ie small slopes. $s \sim \varepsilon$

$q = \nabla^2\psi - F\psi + \beta y$.

- RWs which propogate with shallow water to the right.

Spherical Earth:



Because u predominantly horizontal, only locally vertical component of Ω is relevant so the Coriolis term is $(2\Omega\sin\lambda)\,\hat{z}\wedge\underline{u}$

Suppose our latitude is θ_0..

$$\begin{aligned} f &= 2\Omega\sin\theta \\ &= 2\Omega\sin(\theta_0 + \delta\theta) \qquad \delta\theta \ll 1 \\ &= 2\Omega\sin\theta_0 + 2\Omega\cos\theta_0\,\delta\theta + O(\delta\theta^2) \end{aligned}$$

To a similar level of accuracy (ie $\delta\theta^2$) we can replace the spherical surface by its tangent plane at θ_0.
ie. we take Cartesian coordinates, with O_y poleward, O_x East ward, O_z vertical

Hence $y = (R_E)\delta\theta$ where R_E is the radius of the earth.

Thus $\boxed{f = f_0 + \beta y}$ where $f_0 = 2\Omega\sin\theta_0$ local coriolis parameter
& $\beta = \dfrac{2\Omega\cos\theta_0}{R_E}$

ie, we can take our geometry to be Cartesian and we can take $f = f_0 + \beta y$.
(ROSSBY : β-plane Approximation)
accurate to order $\left(\frac{y}{R_E}\right)^2$

Remember the full SW PV is
$$\begin{aligned} q &= \frac{\zeta + f}{H} \\ &= \frac{\zeta + f_0 + \beta y}{D(1+\varepsilon F\eta)} \qquad \eta_B = 0 \text{ flat bottom ocean.} \\ &= \frac{1}{D}(\zeta + f_0 + \beta y)(1+\varepsilon F\eta)^{-1} \\ &= \frac{1}{D}(\zeta + f_0 + \beta y)\left[1 - \varepsilon F\eta + O(\varepsilon^2)\right] \\ &= \frac{f_0}{D} + \frac{\zeta + \beta y - f_0 F\eta}{D} \qquad D \text{ is a constant} \end{aligned}$$

$\frac{Dq}{Dt} = 0$ becomes

because ζ not nondimensional
$$\frac{D}{Dt}\left(\zeta - f_0 F\eta + \beta y\right) = 0$$

Flat bottomed ocean on rotating sphere.
Identical to channel with sloping bottom
ie. Rossby waves
- change in Coriolis parameter with latitude has same effect as change in depth

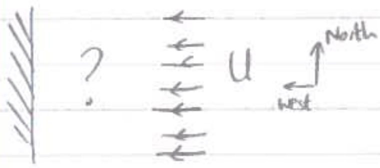
Energy. Long RWs go westward fast.
Short RWs go eastwards slowly.

7/03 $\frac{Dq}{Dt} = 0 \leftarrow$ no linear

QGPV: $q = \nabla^2\psi - F\psi + \eta_B$ $u = -\psi_y$ $v = \psi_x$ $\eta = \psi$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\underline{u}.\underline{\nabla}) = \frac{\partial \cdot}{\partial t} + \frac{\partial(\psi, \cdot)}{\partial(x,y)}$$

Non linear Western Boundary layer (inertial)



Take $\eta_B = \beta y$ (ie. β plane or sloping channel)

Is there a steady flow?
If so what is it?

As $x \to \infty$, $u \to -U$ ie $\psi_y = U$
$v \to 0$ $\psi_x = 0$ as $x \to \infty$

ie $\psi \to U(y - y_0)$ as $x \to \infty$ for some constant y_0

WLOG, We can take $y_0 = 0$ (choosing the origin for y)

Then $\psi \to Uy$ as $x \to \infty$

On $x = 0$, no normal flow so $\psi = \text{const}$. Only one body so w.l.o.g can take $\psi = 0$ on $x = 0$.

BC's: $\psi = 0$ on $x = 0$
$\psi \to Uy$ on $x \to \infty$

Governing equation: $\frac{Dq}{Dt} = 0$ ie. $\left[\frac{\partial}{\partial t} + \frac{\partial(\psi, \cdot)}{\partial(x,y)}\right]\left(\nabla^2\psi - F\psi + \beta y\right) = 0.$

1. q is conserved along particle paths.
But flow steady so p.p's and s'lines coincide
So q is constant along s'lines.

ie. $q = \hat{G}(\psi)$ for some function $\hat{G}$.

or ...

2. $\frac{\partial(\psi, q)}{\partial(x,y)} = 0$ ie. $(\nabla\psi \wedge \nabla q)\cdot\hat{z} = 0$

ie. $\nabla\psi \,//\, \nabla q$

ie. lines of constant ψ // lines constant q.

ie. $q = \hat{G}(\psi)$ for some G.

ie. we have $\nabla^2\psi - F\psi + \beta y = \hat{G}(\psi)$

ie. $\nabla^2\psi + \beta y = \hat{G}(\psi) + F\psi = G(\psi)$

But as $x \to \infty$ $\psi \to Uy$ so $\beta y = G(Uy)$ since $\nabla^2(Uy) = 0$

So for any s, $G(s) = \frac{\beta}{U} s$ $\quad s = Uy$.

Then, $\nabla^2\psi + \beta y = G(\psi) = \frac{\beta}{U}\psi$

Hence $\nabla^2\psi - \frac{\beta}{U}\psi = -\beta y$ - a linear equation
(a linear integral of the non linear equation).

BC's $\psi = 0$, $x = 0$
$\psi \to Uy$, $x \to \infty$.

Write $\psi(x,y) = Uy\,\phi(x)$

Then $\cancel{Uy}\,\phi'' + \beta \cancel{y} = \frac{\beta}{\cancel{U}} \cdot \cancel{Uy}\,\phi$

ie. $\phi'' - \frac{\beta}{U}\phi = -\frac{\beta}{U}$

BC's : $\phi = 0$ on $x = 0$ & $\phi \to 1$ as $x \to \infty$.

A particular solution is $\phi_p = 1$
→ is the flow at infinity

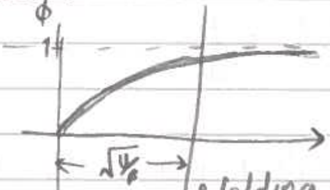
Complementary function $\phi_c = Ae^{-\sqrt{\frac{\beta}{U}}x} + Be^{+\sqrt{\frac{\beta}{U}}x}$

But ϕ bdd as $x \to \infty$ so $B = 0$

Hence GS is $\phi = 1 + Ae^{-\sqrt{\frac{\beta}{U}}x}$

This already satisfies the far-field condition

On the wall, $\phi = 0$ so $A = -1$.

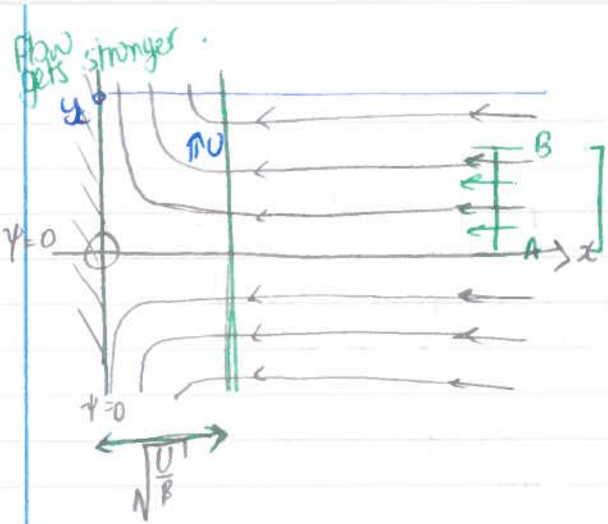


efolding distance for ϕ

ie. $\phi = 1 - e^{-\sqrt{\frac{\beta}{U}}x}$

so. $\psi = Uy\left(1 - e^{-\sqrt{\frac{\beta}{U}}x}\right)$ - a solution of the non linear equation for conservation of QGPV.

efolding scale ⇒ $\sqrt{U/\beta}$



] distance $(y_B - y_A)$ speed U.

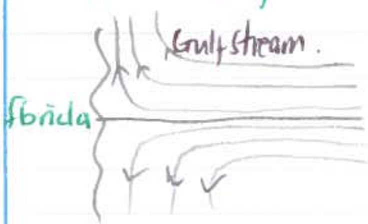
Volume flux across AB as $x \to \infty$ is $U(y_B - y_A)$.
ie. U per unit distance in y direction

The flux across a line $y = y_c$ stretches from wall to ∞, is

$$\int_0^\infty v\,dx = \int_0^\infty \psi_x\,dx = \psi(\infty) - \psi(0) = Uy_c - 0$$

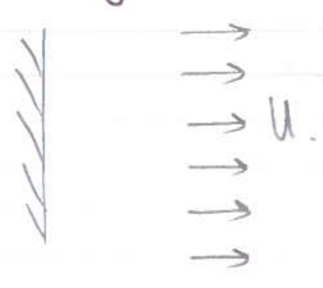
ie. flux in the boundary layer increases at a rate U per unit distance in the y direction.

A Good model for the bottom of the Gulf stream. → as in (southern end)



How about the termination of a WBC? (western Boundary Current)

ie. "Change" $U \to -U$



$$\nabla^2\psi + \frac{\beta}{U}\psi = -\beta y$$

$$\Rightarrow \phi'' + \frac{\beta}{U}\phi = \frac{\beta}{U}$$

Particular solution is 1 : $\phi_p = 1$.

Complementary function: $\phi_c = A\sin\sqrt{\frac{\beta}{U}}\,x + B\cos\sqrt{\frac{\beta}{U}}\,x$

But as $x \to \infty$, $\phi \neq 1$ for any A, B not identically zero.

$\therefore$ there is no boundary layer of this form.

Similarly, there is no solution for the interaction of an Eastern Boundary current $(\beta \to -\beta)$

Rossby waves: $\frac{\partial q}{\partial t} + \frac{\partial(\psi, q)}{\partial(x,y)} = 0.$ $\quad \eta_B = \beta y$ Beta-plane
$q = \nabla^2\psi - F\psi + \beta y.$

$$\frac{\partial}{\partial t}(\nabla^2\psi - F\psi) + \frac{\partial(\psi, \nabla^2\psi + \beta y)}{\partial(x,y)} = 0$$

$$\frac{\partial(\psi, \psi)}{\partial(x,y)} = \psi_x\psi_y - \psi_y\psi_x = 0$$

$$\frac{\partial(\psi, G(\psi))}{\partial(x,y)} = \psi_x G'(\psi)\psi_y - \psi_y G'(\psi)\psi_x = 0$$

Notice a single Rossby wave satisfies this non-linear equation.

Try $\psi = A\cos(kx + ly - \omega t)$

$$\nabla^2\psi = -(k^2+l^2)A\cos(kx+ly-\omega t) = -(k^2+l^2)\psi.$$

For RW $\frac{\partial(\nabla^2\psi, \psi)}{\partial(x,y)} = 0$

Left with linear equation $[+\omega[-k^2-l^2-F] - \beta k]A\sin(kx+ly-\omega t) = 0$

Thus $\omega = \frac{-\beta k}{(k^2+l^2+F)}$ ie. RW's as expected.

$$(c_p)_x = \frac{\omega}{k} = -\frac{\beta}{k^2+l^2+F} < 0 \quad \forall k, l$$

$$(c_p)_y = \frac{\omega}{l} = -\frac{\beta k/l}{k^2+l^2+F}$$

Finite amplitude wave equation.

Stationary phase - max. contribution associated with any wave number travels at a constant speed, the group velocity, $\underline{c}_g = \underline{\nabla}_k \omega$

This is the speed that the wave energy travels at.

Conservation relations.

Physics deals in conservation laws: $\boxed{\dfrac{\partial E}{\partial t} + \underline{\nabla} \cdot \underline{S} = 0}$ → any conservation law can be written like this.

E is the density of the quantity of interest and $\underline{S}$ is the flux of the quantity of interest.

Integrate over a volume V, surface S.

$$\int_V \left(\frac{\partial E}{\partial t} + \underline{\nabla} \cdot \underline{S}\right) dV = 0$$

ie. $\dfrac{d}{dt}\displaystyle\int_V E\,dV + \int_S \underline{S} \cdot \underline{n}\, dS = 0$ by Divergence theorem.

ie. $\dfrac{d}{dt}\displaystyle\int_V E\,dV = -\int_S \underline{S} \cdot \underline{n}\, dS$ outward flux of $\underline{S}$ across S.

ie. The rate of decrease of our quantity = flux out across body.

There is a very important subclass. If $\underline{S} = E\underline{V}$ in some rational way, then we say that our quantity travels at the speed $\underline{V}$ ie. if flux = speed x density.

eg. Conservation of mass: Mass density: density ρ

Cons. of mass eqn: $\dfrac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u}) = 0$

So $E = \rho$ $\underline{S} = \rho \underline{u} = E\underline{u}$

So mass is carried at speed $\underline{u}$.

Rossby wave energy propogation

We only need the linearised QGPV equations.

ie. $\dfrac{\partial}{\partial t}(\nabla^2 \psi - F\psi) + \beta \dfrac{\partial \psi}{\partial x} = 0$

Energy is a quadratic quantity, so multiply by ψ.

$$\psi \frac{\partial}{\partial t} \nabla^2 \psi - F\psi\psi_t + \beta\psi\psi_x = 0$$

ie. $\psi \nabla^2 \psi_t - \left(\tfrac{1}{2}F\psi^2\right)_t + \left(\tfrac{1}{2}\beta\psi^2\right)_x = 0$

$$\left(\psi \underline{\nabla} \cdot (\underline{\nabla}\psi_t) = \underline{\nabla} \cdot (\psi \underline{\nabla}\psi_t) - \underline{\nabla}\psi \cdot \underline{\nabla}\psi_t = \underline{\nabla} \cdot (\psi \nabla \psi_t) - \frac{\partial}{\partial t}\left(\tfrac{1}{2}|\underline{\nabla}\psi|^2\right)\right)$$

ie. $\nabla \cdot (\psi \underline{\nabla} \psi_t) - \dfrac{\partial}{\partial t}\left(\tfrac{1}{2}|\underline{\nabla}\psi|^2\right) - \left(\tfrac{1}{2}F\psi^2\right)_t + \nabla \cdot \left(\tfrac{1}{2}\beta\psi^2 \hat{\underline{x}}\right) = 0$.

($\underline{u}|u|^2$ ie KE/unit mass) ← potential energy/unit mass. as $\psi = \eta$.

(Never unique) Thus $E = \tfrac{1}{2}|\underline{\nabla}\psi|^2 + \tfrac{1}{2}F\psi^2$ (Energy density for a rossby wave)

$\underline{S} = -\psi \nabla \psi_t - \tfrac{1}{2}\beta\psi^2 \hat{\underline{x}}$ ⇒ (non isotropic) → ie. it depends on our coordinate system.

↳ (Energy flux for a Rossby Wave)

NB: Froude number: $\left(\frac{u}{\sqrt{gh}}\right)^2 = \frac{u^2}{gh} \doteq \frac{KE}{PE}$ which seems to be the opposite of F in previous equations
Large Froude $\Rightarrow$ large KE (Possibly an error in his notes?)
Small Froude $\Rightarrow$ large PE $\therefore$ ignore this for now

We will now see that if we can write $\underline{S} = \underline{V}E$ in some sensible way. because then we can say RW energy travels at speed $\underline{V}$.

Our wave is $\psi = A\cos(kx + ly - \omega t)$ $\qquad \theta = kx + ly - \omega t$

so $\psi_x = -kA\sin\theta$
$\psi_y = -lA\sin\theta$.

Thus energy density $E = \frac{1}{2}(k^2+l^2)A^2\sin^2\theta + \frac{1}{2}FA^2\cos^2\theta$.

We need to introduce the idea of an average $\langle E \rangle$.
- the average
- irrelevant if we average over a wavelength in x or a period in t

Choose t. $\langle E \rangle = \frac{\omega}{2\pi}\int_{t_0}^{t_0+\frac{2\pi}{\omega}} E(t)\, dt$ $\qquad \langle h \rangle = \frac{1}{T}\int_0^T h(t)\, dt$ if h has peak T.

Linear operator. $\langle \alpha h_1 + \beta h_2 \rangle = \alpha\langle h_1 \rangle + \beta\langle h_2 \rangle$

Aside: $\langle \cos\theta \rangle = 0$
$\langle \sin\theta \rangle = 0$
$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \langle\cos^2\theta\rangle + \langle\sin^2\theta\rangle = 1$ $\qquad$ But $\langle\cos^2\theta\rangle = \langle\sin^2\theta\rangle$ ($\uparrow$ translated.)
$2\langle\cos^2\theta\rangle = 1$
$\langle\cos^2\theta\rangle = \frac{1}{2}$, $\langle\sin^2\theta\rangle = \frac{1}{2}$

Thus $\langle E \rangle = \frac{1}{4}A^2(k^2+l^2+F)$

$$\underline{S} = -\psi\nabla\psi_t - \frac{1}{2}\beta\psi^2\hat{\underline{x}}$$
$$= -A\cos\theta\,[+k\omega\hat{\underline{x}} + l\omega\hat{\underline{y}}]A\cos\theta - \frac{1}{2}\beta A^2\cos^2\theta\,\hat{\underline{x}}$$
$$= A^2(-\omega\underline{k} - \tfrac{1}{2}\beta\hat{\underline{x}})\cos^2\theta$$

$$\langle \underline{S} \rangle = \frac{1}{2}A^2(-\omega\underline{k} - \tfrac{1}{2}\beta\hat{\underline{x}}).$$
$$= \frac{1}{2}A^2\left(\frac{\beta k}{k^2+l^2+F}(k\hat{\underline{x}} + l\hat{\underline{y}}) - \frac{1}{2}\beta\hat{\underline{x}}\right)$$
$$= \frac{2\langle E \rangle}{(k^2+l^2+F)^2}\left[\beta k^2\hat{\underline{x}} + \beta kl\hat{\underline{y}} - \tfrac{1}{2}\beta(k^2+l^2+F)\hat{\underline{x}}\right]$$

$$= \frac{\beta \langle E \rangle}{(k^2+l^2+F)} \left[(k^2 - l^2 - F)\hat{x} + 2kl\hat{y} \right]$$

Remember $\omega = \frac{-\beta k}{(k^2+l^2+F)}$

$$\frac{\partial \omega}{\partial k} = \frac{-\beta(k^2+l^2+F) + 2\beta k^2}{(k^2+l^2+F)^2} = \frac{\beta(k^2-l^2-F)}{(k^2+l^2+F)^2}$$

$$\frac{\partial \omega}{\partial l} = \frac{2\beta k l}{(k^2+l^2+F)^2}.$$

$$= \langle E \rangle \left[\frac{\partial \omega}{\partial k}\hat{x} + \frac{\partial \omega}{\partial l}\hat{y} \right] = \langle E \rangle \underline{c}_g.$$

Thus $\langle \underline{S} \rangle = \langle E \rangle \underline{c}_g$

ie. the Rossby wave energy travels at the group velocity.

example of this is Rossby wave reflection.

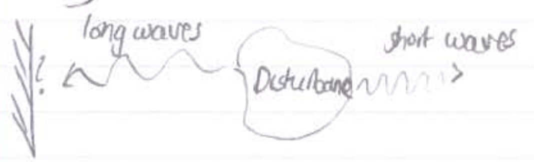
11/03. Phase $e^{i(kx + ly + mz - \omega(k,l,m)t)}$

Static phase : $\frac{\partial}{\partial k}(\text{phase}) = 0$

$$\frac{\partial}{\partial l}(\quad) = 0$$

$$\frac{\partial}{\partial m}(\quad) = 0$$

Rossby wave reflection



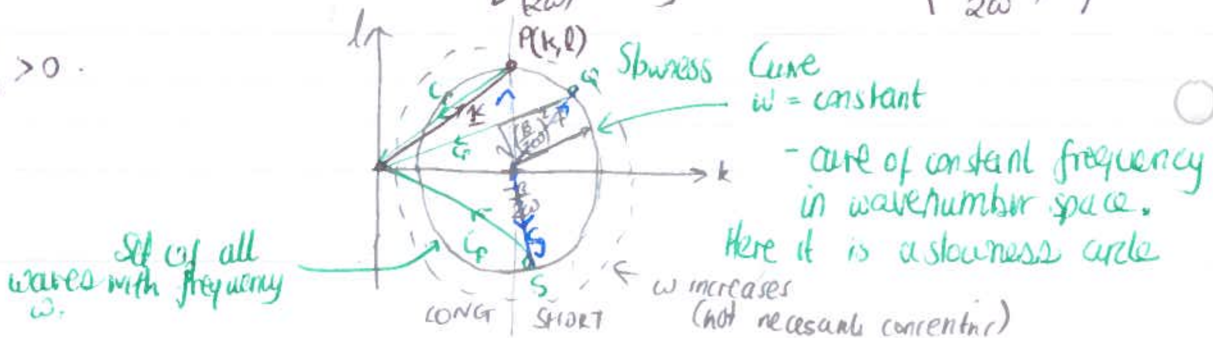
The dispersion relation is $\omega(k,l) = \frac{-\beta k}{k^2+l^2+F}$ $\quad k>0 \Rightarrow \omega < 0.$

ie. $k^2+l^2+F+\frac{\beta}{\omega}k = 0$

$$\left(k+\frac{\beta}{2\omega}\right)^2 + l^2 = \left(\frac{\beta}{2\omega}\right)^2 - F.$$

Circle in (k,l) space with radius $\left[\left(\frac{\beta}{2\omega}\right)^2 - F\right]^{\frac{1}{2}}$ and centre $\left(-\frac{\beta}{2\omega}, 0\right)$

$-\frac{\beta}{2\omega} > 0.$



$$\omega(k,l) = \frac{-\beta k}{k^2+l^2+F}.$$

The phase velocity is by definitle $\underset{\sim}{c}_p = \frac{\omega}{|\underset{\sim}{k}|}\hat{\underset{\sim}{k}}$.

But here $\omega < 0$, so c_p is in direction of $-\underset{\sim}{k}$, where $\underset{\sim}{k} = k\hat{\underline{x}} + l\hat{\underline{y}} = (k,l)$.

Notice c_p always has restricted component since $\frac{\omega}{k} < 0$

- What about c_g?

$\underline{c}_g = \nabla_{\underset{\sim}{k}}\,\omega$
Gradient is always $\perp$'r to the level surfaces
Hence c_g $\perp$'r to slowness surfaces
Thus, here c_g must lie along the radii.

Also $\underline{\nabla}$ is always in the direction of increasing function values.

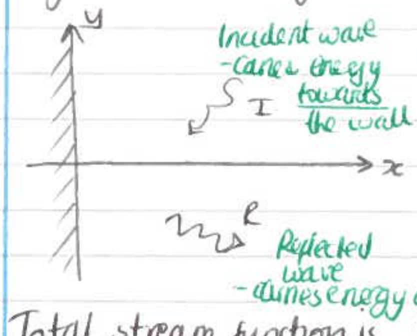
$$\left(k+\frac{\beta}{2\omega}\right)^2 + l^2 = \sqrt{\left(\frac{\beta}{2\omega}\right)^2 - F}.$$

If we increase ω, the radius increases.

$\omega < 0$, so increasing ω moves it closer to zero. thus $|\omega|$ decreases. Thus $\frac{\beta^2}{4\omega^2}$ increases

Energy travels to E if $k > -\frac{\beta}{2\omega}$ short waves and travels to W if $k < -\frac{\beta}{2\omega}$ long waves

[Lighthill Waves generated by travelling forcing effects] (A paper on this).



Given Incident RW
$$\psi_I = \mathrm{Re}\{A_I e^{i(k_I x + ly - \omega_I t)}\}$$

Find the reflected waves
$$\psi_R = \mathrm{Re}\{A_R e^{i(k_R x + l_R y - \omega_R t)}\}$$

Total stream function is $\Psi = \psi_I + \psi_R$

Boundary condition $\Psi = 0$ at $x=0$, $\forall y, \forall t$

At $x=0$, $y=0$ we have $\Psi = 0\ \forall t$

Sufficient to have $A_I e^{-i\omega_I t} + A_R e^{-i\omega_R t} = 0 \quad \forall t$

ie. $e^{i(\omega_I - \omega_R)t} = \underbrace{-\frac{A_I}{A_R}}_{\text{constant.}}$ Thus $\omega_R = \omega_I$

Thus k_R and k_I lies on the same slowness circle : $\omega = \omega_I$. (given).

[If the problem is independant of time, ω is conserved (this is really just energy conservation)].

Also $A_R = -A_I$

Now consider the whole wall. We have (on $x = 0$)

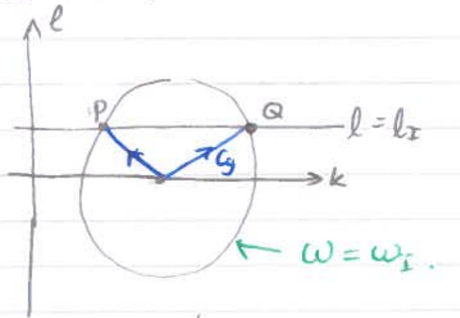
$$A_I\, e^{i(l_I y - \omega_I t)} + A_R\, e^{i(l_R y - \omega_R T)} = 0 \quad \forall y$$

ie. $e^{i l_I y} - e^{i l_R y} = 0 \quad \forall y$

ie $e^{i(l_I - l_R) y} = 1 \ \forall y$ so $l_I = l_R$.

Hence l_I and l_R lie on the same line : $l = l_I$ (given)

(problem is independant of y so y momentum conserved so y wavenumber conserved)



Only 2 waves satisfy these two relations. Thus one is the incident wave ψ_I and the other is the reflected wave ψ_R.

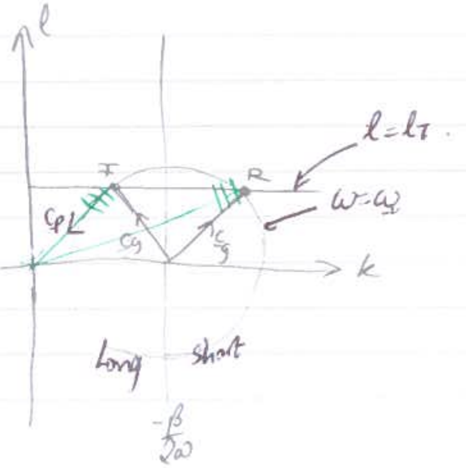
Thus P is the incident wave, carrying energy to the wall and Q is the reflected wave carrying energy ~~to the~~ away from wall.

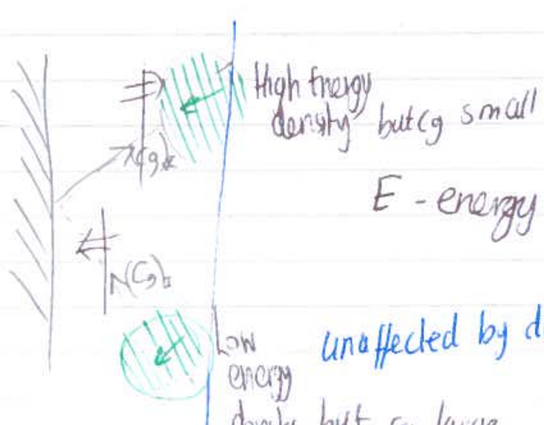
From an Eastern boundary, Q is incident and P is reflected.

Northern/southern
$\therefore k_I, k_R$ lie on $k = k_I$

Arbitrary?
Orthogonal line

14/03.





E - energy density.

unaffected by dissipation

Net energy flux must be zero.

$\langle E \rangle = \frac{1}{4}A^2(k^2+l^2+F)$

$l_R = l_I$

$\frac{k}{2} > -\frac{\beta}{2\omega} > k_I$

$\langle E \rangle_R > \langle E \rangle_I$

Statement about energy flux.

Conservation of energy says that rate at which energy goes inwards = rate at which energy goes outwards.

Product of c_g and $\langle E \rangle$ is the same for incident and reflected waves.

Same in NH + SH (but not at equator where $f=0$)

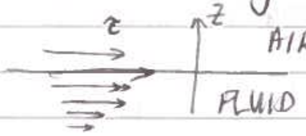
Chapter 3: Dissipation, viscosity and Ekmann layers.

Assume navier stokes

Non rotating frame $\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2 \underline{u}$

$\nu = \frac{\mu}{\rho}$ coefficient of kinematic viscosity which relates this applied stress τ to the rate at which fluid moves.

The final term gives diffusion of momentum with ν, the kinematic viscosity.



$\tau = \mu \frac{\partial u}{\partial z}$ (1D)

or

$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

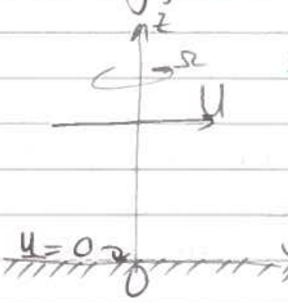
The exta boundary is that not only does $\underline{u}.\underline{\hat{n}} = 0$ on a solid wall but also the tangential component of $\underline{u}$ vanishes ie. $\underline{u} = 0$ on a solid boundary.
no slip condition.

- Relative to a rotating frame; $\frac{D\underline{u}}{Dt} + 2\underline{\Omega} \wedge \underline{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \underline{u}$

  $\nabla . \underline{u} = 0$

  BC $\underline{u} = 0$ on solid boundaries.

In our rotating frame:



$\underline{u} \to U\underline{x}$ as $z \to \infty$

Consider a flow which at large distance is uniform with speed U in the Ox direction.

$u \to U$, $v \to 0$ $z \to \infty$

Our frame is rotating at angular velocity Ω about Oz.

BC: $\underline{u} \to U\hat{\underline{x}}$ as $z \to \infty$
$\underline{u} = 0$ on $z = 0$.

- no approximations made! (as true for honey and water/air)

Boundary conditions are steady so look for solutions independant of t
But BCs are also independant of y so look for solutions independant of y.
" " x " " x

Hence $\underline{u} = \underline{u}(z)$

Continuity: $\cancel{\frac{\partial u}{\partial x}} + \cancel{\frac{\partial v}{\partial y}} + \frac{\partial w}{\partial z} = 0$

But $w=0$ at $z=0$ (no normal flow)

So $w=0$ everywhere. ie. $\forall z$.

Now consider $\frac{Du}{Dt} = \cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} + w\frac{\partial u}{\partial z}$ $\quad w=0$.

Thus $\frac{Du}{Dt} \equiv 0$. Hence the momentum equations are $2\underline{\Omega} \times \underline{u} = -\frac{1}{\rho}\nabla p + \nu\frac{d^2 \underline{u}}{dz^2}$

derivative wrt z.

ie. $-2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu u''$ (x momentum) ①

$2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu v''$. (y momentum) ②

z-momentum: $0 = \frac{\partial p}{\partial z}$

ie. the pressure is independant of the height z

At large z, (1) says $0 = -\frac{1}{\rho}\frac{\partial p}{\partial x}$ so $\frac{\partial p}{\partial x} = 0 \ \forall z$.

As $z \to \infty$ in (2), $2\Omega U = -\frac{1}{\rho}\frac{\partial p}{\partial y}$

So $-\frac{1}{\rho}\frac{\partial p}{\partial y} = 2\Omega U \quad \forall z$

Thus, $\forall z$, $-2\Omega v = 0 + \nu u''$ ①

$2\Omega u = 2\Omega U + \nu v''$ ②

Hence, $\nu u'' + 2\Omega v = 0$ (3)

$\nu v'' - 2\Omega u = -2\Omega U$ (4)

2 linear constant coefficient ODE's with 2bc's each
- we can solve this

BC's $u \to U \quad v \to 0$ as $z \to \infty$

$u = 0 \quad v = 0$ on $z=0$.

The best way of solving this is to introduce the complex velocity $q = u + iv$

(3) + i(4) gives $\nu(u'' + iv'') + 2\Omega(v - iu) = -2\Omega U i$

i.e. $\nu q'' - 2\Omega i q = -2\Omega i U$ → 1 second order constant coefficient ODE with 2BC's.

BC's $q = 0$ on $z = 0$
$q \to U$ as $z \to \infty$

A particular solution is $q_p = U$ (far field is a solution)

To find the complementary function, Try $q = e^{\lambda z}$ and get the auxiliary equation:

$\nu\lambda^2 - 2\Omega i = 0$

i.e. $\lambda^2 = \left(\frac{2\Omega}{\nu}\right)i \Rightarrow \lambda = \pm\sqrt{\frac{\Omega}{\nu}}(1+i)$.

General solution is: $q = U + Ae^{\sqrt{\frac{\Omega}{\nu}}(1+i)z} + Be^{-\sqrt{\frac{\Omega}{\nu}}(1+i)z}$.

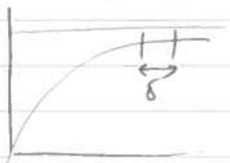
BC's: $q \to U$ as $z \to \infty$ $(\Rightarrow A = 0)$
$q = 0$ on $z = 0$ $(\Rightarrow B = -U)$

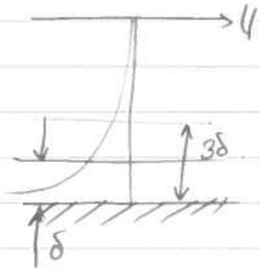
$$\therefore \quad q = U\left[1 - e^{-\sqrt{\frac{\Omega}{\nu}}(1+i)z}\right]$$

E-folding height $\delta = \sqrt{\frac{\nu}{\Omega}}$

Every time z increases by δ, deviation from U decreases by $\frac{1}{e}$.

Boundary layer thickness is δ





EKMAN LAYER.

For water in a container, rotating once per second :- $\nu = 0.01\,cm^2sec^{-1}$.
$\Omega = 2\pi$ radians per sec

$\delta = \sqrt{\frac{0.01\,cm^2sec^{-1}}{6\,sec^{-1}}} = 0.1 \times 0.4\,cm = \frac{1}{2}mm$

$\overrightarrow{\ U\ }$

$\delta = \sqrt{\frac{\nu}{\Omega}}$

$\Omega = 0$:

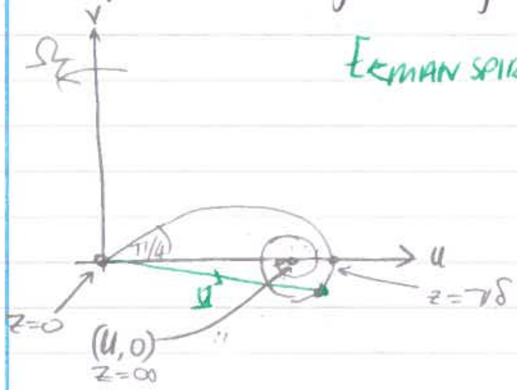
$\Omega \to 0$
$\delta \to \infty$

$q = u + iv$ so $u = \mathrm{Re}\, q$ $\quad v = \mathrm{Im}\, q$

so $u = U\left[1 - e^{-z/\delta}\cos\left(\frac{z}{\delta}\right)\right]$ $\qquad v = Ue^{-z/\delta}\sin\left(\frac{z}{\delta}\right)$

The answer is the same at each x, y but changes with height

Thus plot u as a function of z in the hodograph plane



If $0 < z \ll 1$ $\quad q = U\left[1 - e^{-\frac{z}{\delta}(1+i)}\right]$

$= U\left[1 - \left(1 - \frac{z}{\delta}(1+i)\right)\right]$

$z = \pi\delta$ $\quad v = 0$ $\quad u = U(1 + e^{-\pi})$

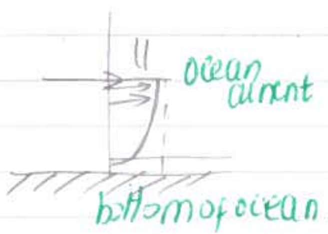
$z = 2\pi\delta$ $\quad v = 0$ $\quad u = U(1 - e^{-2\pi})$

NB: It does not say that the fluid spirals into a point

Not in notes The flux of fluid in this bl is given by

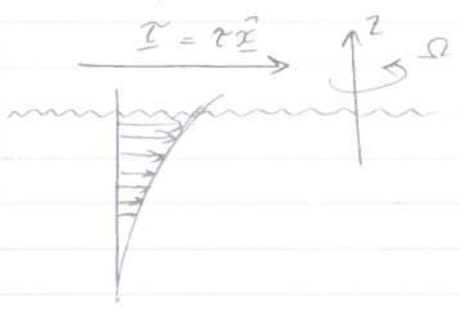
$$\mathcal{F} = \int_{z=0}^{z=\infty} \rho q \, dz = \underline{\underline{\infty}}$$ because if you go up high enough, everything goes at speed U.

The hed ignore as no significant result.



(bottom Ekman bl - benthic bl).

Ocean surface Ekman



Governing equations: $\frac{D\underline{u}}{Dt} + 2\underline{\Omega} \wedge \underline{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{u}$

$\nabla \cdot \underline{u} = 0$

$\underline{u} \to 0$ as $z \to \infty$

What is surface BC?

Now $\tau_{zx} = \tau_0$ $\qquad \tau_{zy} = 0$

z plane $\leftarrow$ x direction

$$\boxed{\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$$ Stress - Strain Relation.

$\tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \qquad$ on $z=0$.

BC's same $\forall t$ so take $\frac{\partial}{\partial t} \equiv 0$

$\forall y \qquad \frac{\partial}{\partial y} \equiv 0$

$\forall x \qquad \frac{\partial}{\partial x} \equiv 0$

Thus $\underline{u} = \underline{u}(z)$.

Thus continuity is $\cancel{\frac{\partial u}{\partial x}} + \cancel{\frac{\partial v}{\partial y}} + \frac{\partial w}{\partial z} = 0$

Hence $\frac{\partial w}{\partial z} = 0$ But $w \to 0$ as $z \to -\infty$ so $w \equiv 0$.

Then $\frac{D\underline{u}}{Dt} = \cancel{\frac{\partial \underline{u}}{\partial t}} + u\cancel{\frac{\partial \underline{u}}{\partial x}} + v\cancel{\frac{\partial \underline{u}}{\partial y}} + \underbrace{\overset{w=0}{\cancel{w}}\frac{\partial \underline{u}}{\partial z}}$

this term vanishes to although $\underline{u}$ is not independant of z.

As before, $2\Omega \wedge \underline{u} = -\frac{1}{\rho}\nabla p + \nu\frac{\partial^2 \underline{u}}{\partial z^2}$

BC's: $\underline{u} \to 0$ as $z \to -\infty$

$\frac{du}{dz} = \frac{\tau_0}{\mu} \quad \frac{dv}{dz} = 0$ on $z=0$.

Notice z-momentum is $0 = \frac{\partial p}{\partial z}$ ie. pressure field is independant of z

We have $-2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu u''$

$2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu v''$

As $z \to -\infty$, $u \to 0$, $v \to 0$ so $\frac{\partial p}{\partial x} = 0 \quad \frac{\partial p}{\partial y} = 0$ at $z = -\infty$

Hence $\frac{\partial p}{\partial x} = 0$, $\frac{\partial p}{\partial y} = 0 \quad \forall z$ since the pressure field is the same for all z.

So we have $-2\Omega v = \nu u''$ ① subject to $u \to 0$, $v \to 0$, $z \to -\infty$

$2\Omega u = \nu v''$ ② $\qquad u' = \frac{\tau_0}{\mu} \quad v' = 0$ at $z=0$.

Introduce $q = u + iv$

① + i② : $\nu q'' = 2\Omega(iu - v) = 2\Omega i q$

BCs: $q \to 0$ as $z \to -\infty$
$q' = \frac{\tau_0}{\mu}$ at $z = 0$

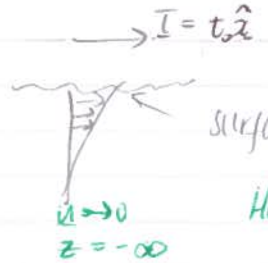
A.E: $\nu\lambda^2 = 2\Omega i$
$\lambda = \pm\sqrt{\frac{\Omega}{\nu}}(1+i)$.

Thus $q = Ae^{(1+i)z/\delta}$ (bdd as $z \to -\infty$) $\boxed{\delta = \sqrt{\frac{\nu}{\Omega}}}$ as before

Then $q'\big|_{z=0} = \frac{A(1+i)}{\delta} = \frac{\tau_0}{\mu}$

Thus $A = \frac{\tau_0\delta}{\mu(1+i)} = \frac{\tau_0\delta}{2\mu}(1-i)$

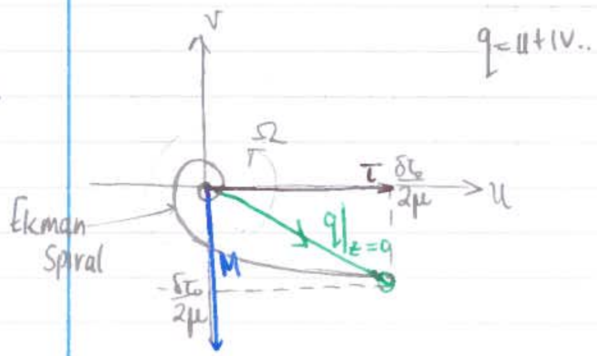
$q = \frac{\tau_0\delta}{2\mu}(1-i)e^{(1+i)z/\delta}$



surface Ekman b.l.

Has finite mass flux, Ekman Spiral.

18/03.



Surface velocity is rotated $\frac{\pi}{4}$ with the direction of rotation.

$q = \frac{\nu}{2\Omega i} q''$ $\quad \nu = \frac{\mu}{\rho}$ kinematic viscosity, coefficient of viscosity

The mass flux per unit area driven by the wind stress is $M = \int_{-\infty}^{0} \rho q \, dz$

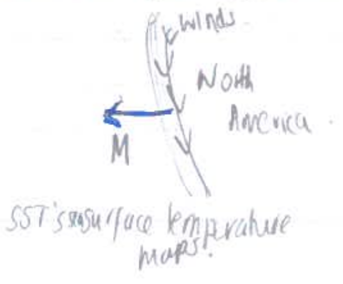
$$= \frac{\rho\nu}{2\Omega i}\int_{-\infty}^{0} q'' \, dz$$

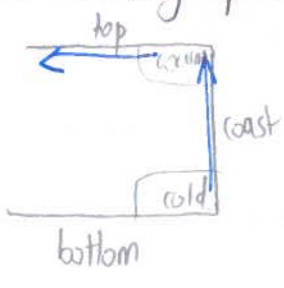
$$= \frac{\mu}{2\Omega i}\left[q'\right]_{-\infty}^{0}$$

$$= \frac{\mu}{2\Omega i} \cdot \frac{\tau}{\mu} = \frac{-\tau_0}{2\Omega i}$$

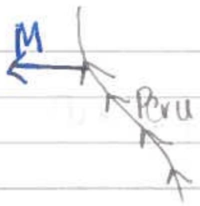
The total mass flux is $\frac{\tau_0}{2\Omega}$ in a direction rotated $\frac{\pi}{2}$ into the rotation from the wind stress dir$^{\underline{n}}$.

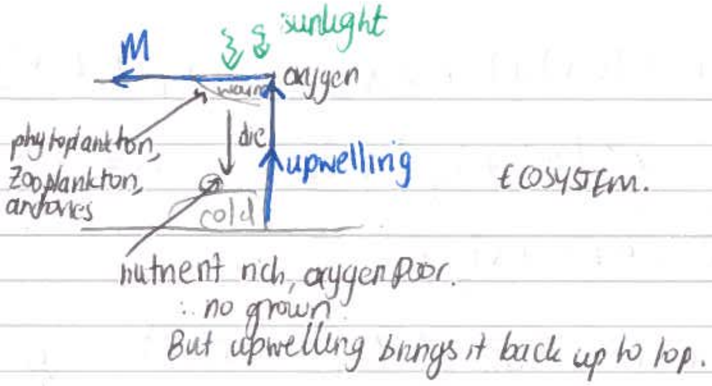
- totally independant of the value of the viscosity
- we do not need to know the viscosity of the ocean to know the Ekman flux





Ekman flux away from coast driven by prevailing winds
- replaced by fluid drawn from the bottom of ocean.



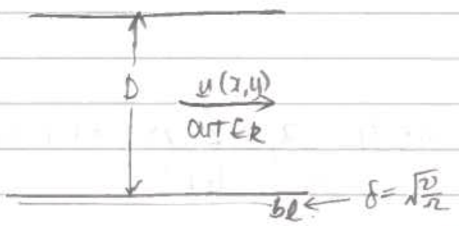


ECOSYSTEM.

Every 4-7 years, the winds collapse - the Ekman flux stops.
- the upwelling stops.

- 3rd Ekman Problem :- Ekman compatibility Condition.

The Ekman layer restricts the outer motion.



Consider a flow of depth D, relative to a rotating frame with the bottom boundary at rest in the rotating frame.

Then $\underline{u}_t + (\underline{u}.\nabla)\underline{u} + 2\underline{\Omega}\wedge\underline{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{u}$

$\nabla.\underline{u} = 0.$ $\quad\quad \underline{u} = 0$ on $z = 0.$

Scale x,y on L
u,v on U
w on $\frac{U}{L}D$
t on L/U
p on $2\Omega\rho UL.$
z on $D.$

$$\cancel{\frac{U}{L}}\frac{\partial u'}{\partial x'} + \cancel{\frac{U}{L}}\frac{\partial v'}{\partial y'} + \cancel{\frac{UD}{LD}}\frac{\partial w'}{\partial z'} = 0$$

$2\underline{\Omega} = 2\Omega\hat{\underline{z}}$

$$\varepsilon(\underline{u}'_{t'} + (\underline{u}'.\underline{\nabla})\underline{u}') + \hat{\underline{z}}\wedge\underline{u}' = -\nabla p' + E\,\nabla^{2'}\underline{u}'$$

Horizontal momentum

$\frac{U}{2\Omega L}$ Rossby no.

$E = \frac{\nu}{2\Omega D^2}$ $\quad\quad \nabla^{2'} = \frac{D^2}{L^2}(\partial^2_{x'} + \partial^2_{y'}) + \partial^2_{z'}.$

We have 3 non dimensional numbers: $\varepsilon = \frac{U}{2\Omega L}$ (Rossby no.)

$E = \frac{\nu}{2\Omega D^2}$ (Ekman no.)

$\frac{D}{L}$ = aspect ration take to be order unity

21/03.

$$\varepsilon(\underline{u}_t + (\underline{u}\cdot\underline{\nabla})\underline{u}) + \hat{\underline{z}} \wedge \underline{u} = -\underline{\nabla} p + E\nabla^2 \underline{u}$$
$\varepsilon = 0$

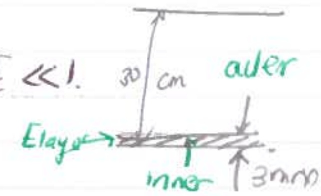
$$\nabla^2 = \Delta^2(\partial_x^2 + \partial_y^2) + \partial_z^2 \qquad \Delta = D/L. \quad (w \text{ scaled on } \Delta U)$$

$$\nabla \cdot \underline{u} = 0$$

$$\underline{u}\cdot\underline{\nabla} = u\partial_x + v\partial_y + w\partial_z$$

Vertical momentum $\Delta^2\varepsilon(w_t + (\underline{u}\cdot\underline{\nabla})w) = -\frac{\partial p}{\partial z} + \Delta^2 E \nabla^2 w.$
$\varepsilon = 0$

Flow is linear $\Rightarrow$ $\underline{\varepsilon = 0}$.
(+ bit steady).



We are interested in the solution for small Ekman number. $E \ll 1$.

$\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \sim 3\text{mm}$ $\qquad \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}/D = \frac{1}{100}$

$D \sim 30\text{cm}$ $\qquad E^{\frac{1}{2}} = 10^{-2} \quad E = 10^{-4}.$

Take $E \to 0$, with all quantities order unity - x, y, z, Δ order unity
Denote this limit by superscript '0's - '0' = 'outer'.

$$\hat{\underline{z}} \wedge \underline{u}^{(0)} = -\nabla p^{(0)}$$

$$\frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} + \frac{\partial w^{(0)}}{\partial z} = 0$$

$$\frac{\partial p^{(0)}}{\partial z} = 0$$

Hence the outer pressure field is independant of z, ie $p^{(0)} = p^{(0)}(x, y)$

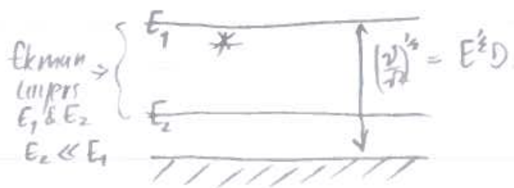
and $u^{(0)} = -\frac{\partial p^{(0)}}{\partial y}$
$v^{(0)} = \frac{\partial p^{(0)}}{\partial x}$ usual geostrophy.

and $\frac{\partial w^{(0)}}{\partial z} = 0.$

ie. the outer flow is geostrophic.
(Different from the first Ekman problem we did, because the outer flow is any geostrophic flow.

BUT
$\underline{u} = 0$ on $z = 0$

need the point (*) to also shrink into E_2
Introduce ζ for this :



$$\zeta = z/E^{\frac{1}{2}}$$

We need to introduce a scaled variable in z so as to retain the viscous terms i.e. the term $E\frac{\partial^2}{\partial z^2}$

Write ζ to be $\frac{z}{E^{1/2}}$: $\zeta = \frac{z}{E^{1/2}}$

$$E^{1/2} d\zeta = dz$$

$$E\frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial \zeta^2}$$

Then we get our boundary equations, Ekman layer equation

$*\zeta \quad \updownarrow (\nu/\Omega)^{1/2} = E^{1/2} D$

$\zeta = \frac{z^*}{DE^{1/2}}$ $\quad z^* \leftarrow$ dimensional z

$z = \frac{z^*}{D}$.

* no superscript -> boundary layer *

$$-V = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial \zeta^2} \quad ①$$

$$u = -\frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial \zeta^2} \quad ②$$

$$0 = \frac{\partial p}{\partial \zeta}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial (E^{-1/2} w)}{\partial \zeta} = 0.$$

For balance in the continuity equation, w is at most $O(E^{1/2})$ so introduce $\tilde{w} = \frac{w}{E^{1/2}}$ to give $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \tilde{w}}{\partial z} = 0$

1) Pressure independant ζ
Thus p same at all ζ. But as $\zeta \to \infty$
We approach the outer flow where $p = p^{(0)}$
Thus in Ekman layer, $p = p^{(0)}$ for all ζ.

$\uparrow \zeta \to \infty$

Then ① becomes $-v = -\frac{\partial p^{(0)}}{\partial x} + \frac{\partial^2 u}{\partial \zeta^2}$ ③

i.e. $-v = -v^{(0)} + \frac{\partial u}{\partial \zeta^2}$ and $u = u^{(0)} + \frac{\partial^2 v}{\partial \zeta^2}$ ④

BC's : $u \to u^{(0)}$ as $z \to \infty$
$v \to v^{(0)}$ as $z \to \infty$
$u^{(0)} = 0$ as $\zeta = 0$, $v^{(0)} = 0$ on $\zeta = 0$.

③ ⇒ define $q = u + iv$.
Then ④ $- i$③ : $u + iv = u^{(0)} + iv^{(0)} + \frac{\partial^2}{\partial z^2}[v - iu]$

ie. $q = q^{(0)} - i q_{\zeta\zeta}$

$$q_{\zeta\zeta} - iq = -iq^{(0)}.$$

PS: $q = q^{(0)}$

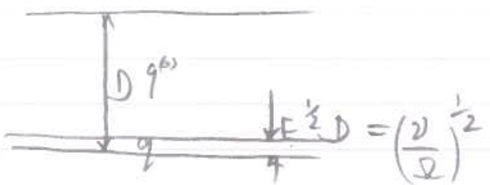
AE $\lambda^2 = i$

$$\lambda = \pm\frac{(1+i)}{\sqrt{2}}$$

Bounded as $\zeta \Rightarrow \infty$ so eliminate $\lambda = +\frac{(1+i)}{\sqrt{2}}$

Hence we have $q = q^{(0)} + Ae^{-\frac{(1+i)\zeta}{\sqrt{2}}}$

But $q = 0$ at $\zeta = 0$ so $q = q^{(0)}\left[1 - e^{-(1+i)\zeta/\sqrt{2}}\right]$ - Ekman Layer Spiral (usual layer thickness proportional to $E^{\frac{1}{2}}$)



In components $u = u^{(0)}\underbrace{\left[1 - e^{-\zeta/\sqrt{2}}\cos(\zeta/\sqrt{2})\right]}_{C} - v^{(0)}\underbrace{\left[e^{-\zeta/\sqrt{2}}\sin(\zeta/\sqrt{2})\right]}_{S}$

$$v = v^{(0)}\left[1 - e^{-\zeta/\sqrt{2}}\cos(\zeta/\sqrt{2})\right] + u^{(0)}e^{-\zeta/\sqrt{2}}\sin(\zeta/\sqrt{2})$$

Then $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial u^{(0)}}{\partial x}C - \frac{\partial v^{(0)}}{\partial x}S + \frac{\partial v^{(0)}}{\partial y}C + \frac{\partial u^{(0)}}{\partial y}S$

$$= \left(\frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y}\right)C - \left(\frac{\partial v^{(0)}}{\partial x} - \frac{\partial u^{(0)}}{\partial y}\right)S$$

But $\frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} = 0$ (outer flow geostrophic)

Also $\frac{\partial v^{(0)}}{\partial x} - \frac{\partial u^{(0)}}{\partial y} = Z^{(0)}$, the vorticity of the outer flow.

Finally, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial \tilde{w}}{\partial \zeta}$

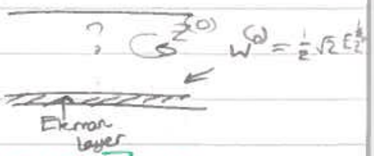
Thus $\frac{\partial \tilde{w}}{\partial \zeta} = Z^{0}e^{-\zeta/\sqrt{2}}\sin(\zeta/\sqrt{2})$ (5).

Thus the vertical velocity in the Ekman layer is non-zero, in general, in contrast to our first example, $u^{(0)} = U$, for which $Z^{(0)} = 0$.

Now integrate (5) through the layer $\int_{\zeta=0}^{\zeta=\infty}\frac{\partial w}{\partial \zeta}\,d\zeta = Z^{(0)}(x,y)\int_0^{\infty} e^{-\zeta/\sqrt{2}}\sin(\zeta/\sqrt{2})\,d\zeta$

$$I = \mathrm{Im}\int_0^\infty e^{-(1-i)\zeta/\sqrt{2}}\, d\zeta.$$
$$= \mathrm{Im}\left(\frac{\sqrt{2}}{1-i}\right)$$
$$= \mathrm{Im}\left(\sqrt{2}(1+i)/2\right)$$
$$= \frac{1}{2}\sqrt{2}$$

outer vorticity drives a weak vertical velocity



$w=0$ on $z=0$

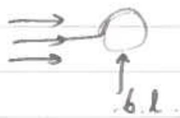
$$\cancel{\tilde{w}(\infty)} - \cancel{\tilde{w}(0)} = \frac{1}{2}\sqrt{2}\, Z^{(0)}$$

$w^{(0)}/E^{\frac{1}{2}}$

Thus $\boxed{w^{(0)} = \frac{1}{2}\sqrt{2}\, E^{\frac{1}{2}} Z^{(0)}}$ EKMAN COMPATIBILITY CONDITION. (DEPENDS ONLY ON THE OUTER QUANTITIES)

The boundary layer imposes a condition on the outer flow (b.l changes the outer flow)

Aside: non rotating



Then the outer flow determines the boundary layer. Not true for a rotating flow

Weak vertical motions generate outer vorticity.

- Variable surface stress $\tau^{(x)}(x,y)\,\hat{\underline{x}} + \tau^{(y)}(x,y)\,\hat{y}$

This drives a vertical velocity of order $E^{\frac{1}{2}}\hat{\underline{z}}\cdot(\underline{\nabla}\cdot\underline{\tau})$ – wind stress curl.

+ve curl ⇓ generate +ve z

−ve curl ⇑ generate −ve z

<u>Ocean circulation</u> – Sverdrup + Stommel b.l –.

Consider the linearised shallow water equations in the form:

$$u_t - fv = -g\eta_x - ru + \tau^{(x)} \quad (1) \qquad r > 0 \text{ positive constant}$$

(includes RAYLEIGH FRICTION)

$$v_t + fu = -g\eta_y - rv + \tau^{(y)} \quad (2) \qquad f = f_0 + \beta y.$$

Get a vorticity equation by taking the curl :- $\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$

$$\zeta_t + f(u_x + v_y) = -r\zeta + \frac{\partial}{\partial x}\tau^{(y)} - \frac{\partial}{\partial y}\tau^{(x)} \qquad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

For geostrophic motion $u = -\psi_y \quad v = \psi_x \qquad \zeta = \nabla^2\psi.$

$$u_x + v_y = 0$$

$$\nabla^2\psi_t + \beta\psi_x = -r\nabla^2\psi + \frac{1}{\rho D}\hat{\underline{z}}\cdot(\underline{\nabla}\wedge\underline{\tau})$$

This equation governs wind forced ocean circulation on a β-plane :- (LINEAR)

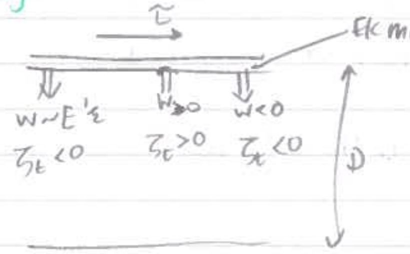
$$\nabla^2 \psi_t + \beta \psi_x = -r \nabla^2 \psi + \frac{1}{\rho D} \hat{z} \cdot (\nabla \wedge \underline{\tau})$$

rate of change of vorticity

effect of gradient planetary vorticity.

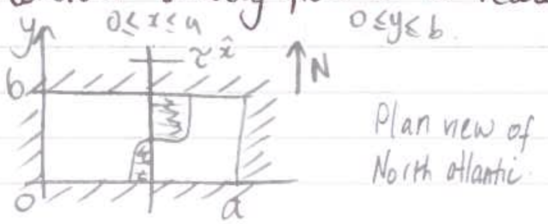
Rayleigh friction

wind stress and forcing (generate vertical velocities at bottom of surface Ekman layer - generates vorticity)



Sverdrup : steady frictionless
- balance between β effect + wind stress curl
- extremely good almost everywhere.

Example: Consider steady flow in a rectangular basin



Let the ocean be forced by the wind stress $\underline{\tau} = -\tau_0 \cos\left(\frac{\pi y}{b}\right) \hat{\underline{x}}$

This gives $\hat{\underline{z}} \cdot \text{curl} \underline{\tau} = \frac{\partial \tau^{(y)}}{\partial x} - \frac{\partial \tau^{(x)}}{\partial y} = -\tau_0 \frac{\pi}{b} \sin \frac{\pi y}{b}$.

Our equation is $\beta \psi_x = -r \nabla^2 \psi - \frac{\pi \tau_0}{\rho D b} \sin \frac{\pi y}{b}$

The bc's are no flow through boundaries

ie. $\psi = 0$ on $x = 0, x = a$
on $y = 0, y = b$.

Now non dimensionalise, $x = ax'$ so $0 \le x' \le 1$
$y = by'$ $\quad 0 \le y' \le 1$

$\psi = \frac{a \pi \tau_0}{\beta D b \rho} \psi'$

$dx = a\,dx'$
$\frac{\partial}{\partial x} = \frac{1}{a} \frac{\partial}{\partial x'}$ $\quad \frac{\partial^2}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2}{\partial x'^2}$

Then (dropping 's) $\boxed{\psi_x = -\varepsilon_s \nabla^2 \psi - \sin \pi y}$. Here $\nabla^2 = \partial_x^2 + \frac{a^2}{b^2} \partial_y^2$.

to solve.

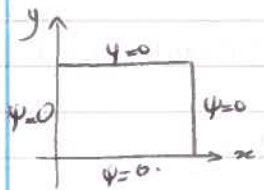
where $\varepsilon_s = \frac{r}{\alpha\beta}$. s = stommel.

BCs: ~~4~~ $\psi = 0$ on $x = 0, 1$
$y = 0, 1$

Need to solve the problem for $0 < \varepsilon_s \ll 1$ (small friction)

Start: zero friction: Sverdrup $\varepsilon_s = 0 \Rightarrow$ singular perturbation problem because ε_s multiplies the highest derivative $\Rightarrow$ layer = <u>Stommel layer</u>

25/03.



β-effect
wind stress $>$ Sverdrup circulation.

$y^* = by$
$x^* = ax$

$$\psi_x = -\varepsilon_s \nabla^2 \psi - \sin\pi y \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{a^2}{b^2}\frac{\partial^2}{\partial y^2}$$

$\varepsilon_s = \frac{r}{a\beta}$.

Solve this for $0 < \varepsilon_s \ll 1$
$\psi = 0$ on $x = 0, 1$, and $y = 0, 1$.

Try $\varepsilon_s = 0$.

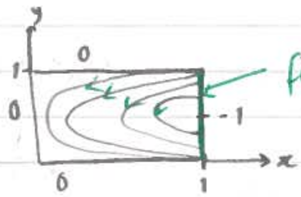
Sverdrup: $\varepsilon_s = 0$

$$\psi_x = -\sin\pi y \qquad \psi = -x\sin\pi y + A(y)$$

Now $\psi = 0$ on $y = 0, 1$, so $A(0) = 0$
and $A(1) = 0$

At $x = 0$, $0 = 0 + A$ so $A = 0$ and hence $\psi = -x\sin\pi y$.

But now $\psi|_{x=1} = -\sin\pi y \neq 0 \ \forall y$.

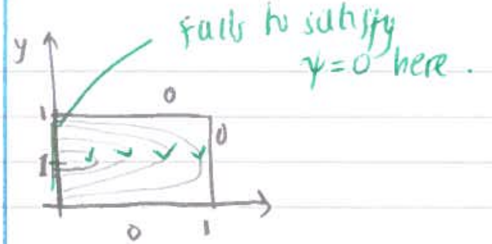


fails to satisfy bc here.

Or at $x = 1$, $\psi = 0$ so $0 = -\sin\pi y + A(y)$

Then $A(y) = \sin\pi y$ $A(0) = 0$ ✓
$A(1) = 0$ ✓

Giving $\psi = (1-x)\sin\pi y$. But now $\psi|_{x=0} = \sin\pi y \neq 0 \ \forall y$.



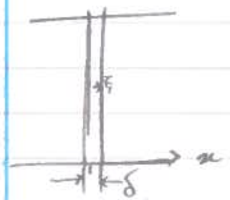
Sverdrup

- Singular perturbation problem.

If you throw out higher derivatives, cannot easily satisfy all bc's.
Need a thin layer where the higher derivatives are important.
No problem with y bc's. Thus the layer is thin in x. How thin?

$\varepsilon_s \nabla^2 \psi + \psi_x = -\sin \pi y$.

Let layer have thickness δ. Write $x = \delta \xi$. (ξ is order unity. $0 < \delta \ll 1$)



$$\frac{\varepsilon_s}{\delta^2} \frac{\partial^2 \psi}{\partial \xi^2} + \cancel{\varepsilon_s \frac{a^2}{b^2} \frac{\partial^2 \psi}{\partial y^2}} + \frac{1}{\delta} \psi_\xi = \cancel{-\sin \pi y}.$$

Notice if $\varepsilon_s = 0$ and $\delta = O(1)$: Sverdrop.

Now let δ go to 0, $\frac{1}{\delta}\psi_\xi$ becomes large

Only other term that can become large is $\frac{\varepsilon_s}{\delta^2} \psi_{\xi\xi}$

ie. these two terms must balance in any thin layer.

Thus $\frac{\varepsilon_s}{\delta^2} = \frac{1}{\delta}$ ie $\delta = \varepsilon_s$.

our gov'ing eq^n

Thus we have a Stommel layer of thickness ε_s with which $\psi_{\xi\xi} + \psi_\xi = 0$

This has solution $(\psi_\xi)_\xi = -\psi_\xi$

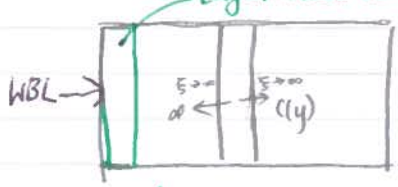
ie. $\psi_\xi = B(y)\, e^{-\xi}$.

ie $\psi_s = C(y) - B(y)\, e^{-\xi}$.
s ← for 'stommel'.

layer must be situated at $x = 0$ so $\xi \geq 0$

Now for $\xi \to +\infty$, $\psi_s \to C(y)$

and as $\xi \to -\infty$, $|\psi_s| \to \infty$



as then, we don't have to worry about ψ going to ∞

Problem! : there is a unique place we can put this layer.

Then on $\xi = 0$, $\psi_{WBL} = 0$ WBL = WESTERN BOUNDARY LAYER.

Thus $B(y) = C(y)$ and $\psi_{WBL} = C(y)\left[1 - e^{-\xi}\right]$

this satisfies the bc $\psi = 0$. (WBL)

The outer solution needs to satisfy the other 3 boundary conditions ie. Sverdrups Solution.

(which was $\psi_{SV} = (1-x)\sin \pi y$

We want our solutions to match smoothly. We require that $\lim_{\xi \to \infty} \psi_{WBL} = \lim_{x \to 0} \psi_{SV}$.

Thus $C(y) = \sin \pi y$.
Thus we have $\psi_{WBL} = \sin \pi y \left[1 - e^{-(x/\varepsilon_s)}\right]$

Here, we can form a composite solution by taking $\psi_{comp} = \psi_{WBL} + \psi_{SV} - (\psi_{SV})\big|_{x=0}$

(same as subtracting the $(\psi_{WBL})_{\xi \to \infty}$).

Then we have $\psi_c = \sin \pi y \left\{1 - e^{-(x/\varepsilon_s)} + 1 - x - 1\right\}$

$$\psi_c = \sin \pi y . \left(1 - x - e^{-x/\varepsilon_s}\right)$$

How good is this solution?

$y = 0, 1$ $\psi_c = 0$
$x = 0$ $\psi_c = 0$
$x = 1$ $\psi_c = -e^{-1/\varepsilon_s} \sin \pi y$.

this is extremely small